Planar traveling waves in capillary fluids

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Abstract

By capillary fluids we mean compressible, inviscid fluids whose energy depends not only on their density but also on their density gradient. Their motion is thus governed by systems of conservation laws, either in Eulerian coordinates or in Lagrangian coordinates, that are higher order modification of the usual equations of gas dynamics. In both settings, we receive models that also arise in other fields, in particular in water waves theory and quantum hydrodynamics. Those Hamiltonian systems typically admit three types of planar traveling waves, namely, heteroclinic, homoclinic, and periodic ones. The purpose here is to review the main tools and results regarding the stability of those waves, under most general assumptions on the energy law. Special attention is devoted to the correspondence between traveling waves in Eulerian coordinates and those in Lagrangian coordinates.

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Contents

1 Introduction

The Euler–Korteweg system of PDEs arises as a mathematical model for various phenomena in fluid dynamics, *e. g.* flow of capillary fluids (liquid-vapor mixtures, superfluids), long water waves, vortex dynamics, quantum hydrodynamics. Its main features are

- conservativity, the equations being local conservation laws,
- reversibility, which means that all possible diffusion processes are neglected - neither viscosity nor heat conductivity is taken into account,
- dispersivity due to higher order derivatives that are not present in the usual Euler equations for compressible fluids,
- and a natural Hamiltonian structure in terms of the total energy (which is conserved because of reversibility).

Mathematical physicists acquainted with, for instance, the Korteweg–de Vries equation (KdV), would suspect another feature from the above ones, namely, integrability. However, we are mostly interested in 'real' fluids, with rather general pressure laws (as well as with general capillarity coefficients, to allow some flexibility when passing from Eulerian coordinates to Lagrangian coordinates). To some extent, the Euler–Korteweg system we consider is a vector-valued analog of the so-called generalized Korteweg–de Vries equation (gKdV)[1]. General nonlinearities as we have preclude algebraic approaches based on integrability. So we will not dwell much on algebraic aspects, even though we do perform some algebra in the sequel.

[1]In fact, we shall point out explicit relationships between the Euler–Korteweg equations and other famous dispersive equations, namely the NonLinear Schrödinger equation (NLS) and the generalized Boussinesq equation.

As expected from a general observation made by Benjamin [4], a Hamiltonian structure is inherited by the ODEs governing the planar travelling waves that are independent of transverse variables (the only ones that we will consider here; otherwise, we would have to deal with elliptic PDEs, which is not our purpose). These ODEs being two-dimensional, their Hamiltonian structure make them integrable by quadrature. Therefore, the existence and classification of the Euler–Korteweg planar travelling waves follows from an easy phase portrait analysis. Trickier is their stability analysis, which can be addressed from several points of view and is the main topic of this survey paper.

2 The Euler–Korteweg system

2.1 Original and related models

The most general form of what we call the Euler–Korteweg system is made of the $(d+1)$ evolution PDEs in space dimension d

$$(1)\qquad \begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) \;=\; 0\,, \\ \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} \;+\; \nabla(\delta\mathscr{E}) \;=\; 0\,. \end{cases}$$

These equations are supposedly governing the motion of an inviscid fluid whose density is ρ, velocity is $\mathbf{u}$, and internal or free[2] energy per unit volume is $\mathscr{E}$. The first equation in (1) is the *continuity equation*, expressing the (local) conservation of mass. The velocity equation is written in terms of $\delta\mathscr{E}$, the *variational gradient* of $\mathscr{E}$. If this energy depends only on ρ then

$$\delta\mathscr{E} \;=\; \frac{\mathrm{d}\mathscr{E}}{\mathrm{d}\rho}\,,$$

and (1) is nothing but the usual Euler equations (in non-conservative form) for compressible fluids. Here we are most interested in 'nonclassical' fluids in which E also depends on $\nabla\rho$. In this case

$$\delta\mathscr{E} \;=\; \mathsf{E}_\rho\mathscr{E} \;:=\; \frac{\partial\mathscr{E}}{\partial\rho} \;-\; \sum_{j=1}^{d} \mathrm{D}_{x_j}\left(\frac{\partial\mathscr{E}}{\partial\rho_{x_j}}\right)\,,$$

where E_ρ stands for the *Euler operator* with respect to ρ (we shall use Euler operators with respect to velocity components u_j later on), D_{x_j} denotes the *total derivative* with respect to x_j while ρ_{x_j} means the partial derivative of ρ with respect to x_j. A typical example that goes back to Korteweg's theory of capillarity is

$$(2)\qquad \mathscr{E}(\rho,\nabla\rho) = F(\rho) \;+\; \frac{1}{2}K(\rho)\,\|\nabla\rho\|^2\,,$$

[2]Physically, $\mathscr{E}$ corresponds to the free energy for isothermal motions, and to the internal energy for adiabatic motions.

for which

$$\delta\mathscr{E} \;=\; F'(\rho) \,+\, \tfrac{1}{2}K'(\rho)\,\|\nabla\rho\|^2 \,-\, \mathrm{div}(K(\rho)\Delta\rho) \tag{3}$$
$$\;=\; g(\rho) \,-\, \frac{1}{2}K'(\rho)\,\|\nabla\rho\|^2 \,-\, K(\rho)\,\Delta\rho\,, \quad g \,:=\, \frac{\mathrm{d}F}{\mathrm{d}\rho}\,.$$

As recalled in Appendix, (1) can be deduced from a variational principle using the 'natural' Lagrangian

$$\frac{1}{2}\,\rho\,\|\mathbf{u}\|^2 \,-\, \mathscr{E}(\rho,\nabla\rho)\,.$$

Moreover, irrotational velocities remain so (at least formally) under evolution by (1), and for *irrotational flows* (1) admits a Hamiltonian structure associated with the total energy

$$\mathscr{H} := \frac{1}{2}\,\rho\,\|\mathbf{u}\|^2 \,+\, \mathscr{E}(\rho,\nabla\rho)\,.$$

Indeed, since

$$\delta\mathscr{H} \;=\; \begin{pmatrix} \tfrac{1}{2}\,\|\mathbf{u}\|^2 \,+\, \delta\mathscr{E} \\ \rho\,\mathbf{u} \end{pmatrix}$$

and

$$\frac{1}{2}\,\nabla(\|\mathbf{u}\|^2) \;=\; (\mathbf{u}\cdot\nabla)\mathbf{u}$$

for irrotational vector fields $\mathbf{u}$, (1) can be viewed as the infinite-dimensional Hamiltonian system

$$\partial_t \begin{pmatrix} \rho \\ \mathbf{u} \end{pmatrix} \;=\; -\begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0_d \end{pmatrix} \delta\mathscr{H}\,. \tag{4}$$

(In fact, (1) admits a Hamiltonian structure associated with $\mathscr{H}$ for more general vector fields $\mathbf{u}$, namely those admitting *Clebsch coordinates*, which play the role of 'canonical variables' in this infinite-dimensional setting, see §A.2 in the appendix for more details.)

Another remarkable feature is that (1) can equivalently be written, as far as smooth solutions are concerned, in the conservative form

$$\left\{ \begin{aligned} &\partial_t\rho + \mathrm{div}(\rho\mathbf{u}) \;=\; 0\,, \\ &\partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) = \mathrm{div}\boldsymbol{\Sigma}\,, \end{aligned} \right. \tag{5}$$

where

$$\boldsymbol{\Sigma} \;:=\; (\mathscr{E} \,-\, \rho\,\delta\mathscr{E})\,I_d \,-\, \nabla\rho\otimes\frac{\partial\mathscr{E}}{\partial\nabla\rho} \tag{6}$$

is the *energy-momentum tensor* (under mass constraint) in connection with Noether's theorem (see for instance [12, p. 121], who unfortunately does not refer to Noether): the d

(local) conservation laws for the momentum $\rho\mathbf{u}$ are linked to the invariance of the energy with respect to translations in space variables x_j, $j = 1, \dots, d$. In coordinates,

$$\Sigma_j^k \,=\, (\mathscr{E} \,-\, \rho\,\delta\mathscr{E})\,\delta_j^k \,-\, \rho_{x_k}\frac{\partial\mathscr{E}}{\partial\rho_{x_j}}\,.$$

The reader not familiar with Noether's theorem will verify with bare hands the identity

$$\frac{1}{\rho}\,\mathrm{div}\boldsymbol{\Sigma} \,+\, \nabla(\delta\mathscr{E}) \,=\, 0\,,$$

and combine the mass conservation law with the velocity equation to get the momentum conservation law[3]. Now, thanks to the conservative form (5) we can easily rewrite the equations in *Lagrangian coordinates*, at least in space dimension one. Indeed, the one-dimensional system

$$\left\{\begin{array}{l}\partial_t\rho + \partial_x(\rho u) \,=\, 0\,,\\[1ex] \partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x\Sigma\,, \quad \Sigma \,:=\, \mathscr{E} \,-\, \rho\,\delta\mathscr{E} \,-\, \rho_x\dfrac{\partial\mathscr{E}}{\partial\rho_x}\,,\end{array}\right. \tag{7}$$

is found to be equivalent to

$$\left\{\begin{array}{l}\partial_s\widetilde{v} - \partial_y\widetilde{u} \,=\, 0\,,\\ \partial_s\widetilde{u} \,=\, \partial_y\widetilde{\Sigma}\,, \quad \widetilde{\Sigma}(\widetilde{v},\widetilde{v}_y,\widetilde{v}_{yy}) \,:=\, \Sigma(\rho,\rho_x,\rho_{xx})\,,\end{array}\right. \tag{8}$$

where y is the *mass Lagrangian coordinate* defined by

$$\mathrm{d}y \,=\, \rho\,\mathrm{d}x \,-\, \rho\,u\,dt\,,$$

where $s = t$ is time, $\widetilde{u}(y,s) = u(x,t)$, and

$$\widetilde{v}(s,y) = \frac{1}{\rho(x,t)}\,,$$

is the specific volume. The existence of y follows from the conservation law of mass in (7). A straightforward way to pass from the conservation law of momentum in (7) to the velocity equation in (8) is to note that both are equivalent (in a simply connected domain) to the differential form

$$\rho\,u\,\mathrm{d}x \,+\, (\Sigma \,-\, \rho u^2)\,\mathrm{d}t \,=\, \widetilde{u}\,\mathrm{d}y \,+\, \widetilde{\Sigma}\,\mathrm{d}s$$

being exact. In the sequel, we omit the tildas when no confusion can occur.

[3]All these computations are made from a purely algebraic point of view (from the analytical point of view we should assume the dependent variables to be smooth enough in order to justify all manipulations of derivatives).

Finally, let us observe that

$$\widetilde{\Sigma} \,=\, \delta\mathscr{e} \,=\, \mathsf{E}_v\mathscr{e} \,=\, \frac{\partial \mathscr{e}}{\partial v} \,-\, \mathrm{D}_y\left(\frac{\partial \mathscr{e}}{\partial v_y}\right) \tag{9}$$

where

$$\mathscr{e}(v, v_y) \,:=\, \frac{1}{\rho}\,\mathscr{E}(\rho, \rho_x) \tag{10}$$

is the specific (internal or free) energy. Indeed, we have by definition

$$\rho \,=\, \frac{1}{v}\,, \quad \rho_x \,=\, -\,\frac{v_y}{v^3}\,,$$

so that

$$\frac{\partial \mathscr{e}}{\partial v} \,=\, \mathscr{E} \,+\, v\,\left(-\frac{1}{v^2}\,\frac{\partial \mathscr{E}}{\partial \rho} \,+\, 3\,\frac{v_y}{v^4}\,\frac{\partial \mathscr{E}}{\partial \rho_x}\right) \,=\, \mathscr{E} \,-\, \rho\,\frac{\partial \mathscr{E}}{\partial \rho} \,-\, 3\,\rho_x\,\frac{\partial \mathscr{E}}{\partial \rho_x}\,,$$

$$\frac{\partial \mathscr{e}}{\partial v_y} \,=\, v\,\left(-\frac{1}{v^3}\,\frac{\partial \mathscr{E}}{\partial \rho_x}\right) \,=\, -\,\rho^2\,\frac{\partial \mathscr{E}}{\partial \rho_x}\,,$$

$$\mathrm{D}_y\left(\frac{\partial \mathscr{e}}{\partial v_y}\right) \,=\, v\,\mathrm{D}_x\left(-\,\rho^2\,\frac{\partial \mathscr{E}}{\partial \rho_x}\right) \,=\, -\,\rho\,\mathrm{D}_x\left(\frac{\partial \mathscr{E}}{\partial \rho_x}\right) \,-\, 2\,\rho_x\,\frac{\partial \mathscr{E}}{\partial \rho_x}\,.$$

In particular when the energy $\mathscr{E}$ is as in (2), we have in one space dimension

$$\mathscr{E}(\rho, \rho_x) = F(\rho) \,+\, \frac{1}{2}K(\rho)\,(\rho_x)^2\,, \tag{11}$$

$$\mathscr{e}(v, v_y) \,=\, f(v) \,+\, \frac{1}{2}\,\kappa(v)\,(v_y)^2\,, \tag{12}$$

with

$$f(v) \,:=\, \frac{1}{\rho}\,F(\rho)\,, \quad \kappa(v) \,:=\, \rho^5\,K(\rho)\,.$$

In all cases, the one-dimensional equations of motion in Lagrangian coordinates (8) read

$$\left\{\begin{aligned} \partial_s v \,&=\, \partial_y u\,, \\ \partial_s u \,&=\, \partial_y(\delta\mathscr{e})\,, \end{aligned}\right. \tag{13}$$

of which the Hamiltonian formulation analogous to (4) is

$$\partial_s\begin{pmatrix} v \\ u \end{pmatrix} \,=\, \begin{pmatrix} 0 & \partial_y \\ \partial_y & 0 \end{pmatrix}\,\delta\mathscr{h}\,, \quad \mathscr{h} \,:=\, \frac{1}{2}\,u^2 \,+\, \mathscr{e}(v, v_y)\,. \tag{14}$$

Indeed, we have

$$\delta\mathscr{h} \,=\, \begin{pmatrix} \delta\mathscr{e} \\ u \end{pmatrix}$$

where $\delta \mathscr{e} = \mathsf{E}_v \mathscr{e}$ as defined in (9), hence the equivalence between (13) and (14). Let us warn the reader that the significance of the symbol δ for the variational derivative $\delta \mathscr{k}$ in (14) is different from the one for $\delta \mathscr{H}$ in (4) because $\mathscr{k}$ is viewed as a function of (v, v_y, u) whereas $\mathscr{H}$ is viewed as a function of (ρ, ρ_x, u).

One purpose of this paper is to shed light on the interplay between the equations in Lagrangian coordinates (13) and those in Eulerian coordinates,

$$\text{(15)} \qquad \left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) \;=\; 0\,, \\ \partial_t u + u \partial_x u \;+\; \partial_x(\delta \mathscr{E}) \;=\; 0\,, \end{array} \right.$$

or equivalently

$$\text{(16)} \qquad \partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} \;=\; - \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \delta \mathscr{H}\,, \quad \mathscr{H} \;=\; \frac{1}{2}\, \rho\, u^2 \;+\; \mathscr{E}(\rho, \rho_x)\,,$$

the one dimensional version of (4), in the analysis of various types of 'localized' solutions.

Related models When the energy has the form (2) with capillarity coefficient

$$K(\rho) = \frac{1}{4\rho}\,,$$

the Euler–Korteweg system (1) equivalently reads

$$\text{(17)} \qquad \left\{ \begin{array}{l} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) \;=\; 0\,, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} \;+\; \nabla g(\rho) \;=\; \nabla \left(\dfrac{\Delta \sqrt{\rho}}{2\sqrt{\rho}} \right)\,, \end{array} \right.$$

which may be viewed as a reformulation *via* the Madelung transform

$$\psi \;\mapsto\; (\rho, \mathbf{u})\,;\; \psi = \sqrt{\rho}\, \exp^{i\phi}\,,\; \mathbf{u} = \nabla \phi\,,$$

of the NonLinear Schrödinger equation (NLS)

$$\text{(18)} \qquad i \partial_t \psi \;+\; \frac{1}{2}\, \Delta \psi \;=\; \psi\, g(|\psi|^2)$$

away from zeroes of ψ. In particular, the Gross-Pitaevskii equation corresponds to $g(\rho) = \rho - 1$.

In the first approximation, the evolution of vortex curves in 3-D incompressible fluids obeys the so-called *filament equation* (see [2, p. 332])

$$\partial_t \gamma \;=\; \partial_x \gamma \times \partial_x^2 \gamma\,,$$

where $\gamma = \gamma(x, t)$ is a parametrization of the curve by $x \in S^1$ at time t. Equivalent reformulations of this equation are the 1-D NonLinear Schrödinger equation (18) with

$g(\rho) = -\rho/4$ (a focusing case!), or the 1-D Euler–Korteweg system (15) where the energy is as in (11) with

$$K(\rho) = \frac{1}{4\rho}\,,\ F(\rho) \,=\, -\frac{\rho^2}{8}\,,$$

and $\rho(x,t) = k(x,t)^2$ where $k(x,t)$ is the curvature of γ while $u(x,t)$ plays the role of its torsion.

In water waves theory, the first nonlinear and dispersive equation was derived by Boussinesq [16], and reads

$$\partial_t^2 h - \, g\, H\, \partial_x^2 \left(h \,+\, \frac{3\, h^2}{2\, H} \,+\, \frac{H^2}{3}\, \partial_x^2 h \right) \,=\, 0\,,$$

where g denotes the gravity, H is the reference depth of water, and h is the height of water waves. After being forgotten for almost one century, this equation came back to light in a generalized form

$$\partial_t^2 h \,+\, \partial_x^2 p(h) \,+\, \kappa\, \partial_x^4 h \,=\, 0\,, \tag{19}$$

considered in particular by Bona and Sachs [14], together with its 2×2 system version

$$\left\{ \begin{array}{l} \partial_t h \,=\, \partial_x u\,, \\ \partial_t u \,+\, \partial_x p(h) \,=\, -\,\kappa\, \partial_x^3 h\,. \end{array} \right. \tag{20}$$

Changing notations in (20), and more precisely substituting s for t, v for h, and $-f'$ for p, we recognize the Euler–Korteweg system in Lagrangian coordinates (13) with the energy as in (12). Another way of inviting the Euler–Korteweg system in water waves theory is to take into account *surface tension*, which is meaningful for ripples over thin films. This amounts to adding a third order term to *Saint Venant*'s shallow water equations (see [17, 18] for more details), and leads to the Euler–Korteweg system in Eulerian coordinates (15) in which the energy is as in (11) with F being quadratic.

2.2 Gallery of traveling waves

Limiting ourselves to *planar* traveling waves that are independent of transverse variables (for more complicated patterns, see for instance [13] and related work), we need only consider the one-dimensional equations (15) to find such waves. In this respect, it is tempting to consider the simpler-looking equations in Lagrangian coordinates (13). However, if we have in mind multidimensional perturbations it seems important to stick to Eulerian coordinates (because Lagrangian coordinates in several space dimensions are hardly handleable). So we shall keep track of both, and on the way we shall pay attention to the relationship between Lagrangian traveling waves and Eulerian traveling waves. First of all, let us emphasize that the 'speed' of a Lagrangian traveling wave is actually not a speed but a momentum density. Nevertheless, the governing ODEs of both Lagrangian traveling waves and Eulerian traveling waves have the common feature of being Hamiltonian, as predicted by [4] for general Hamilonian PDEs. A little more subtle is the one-to-one correspondence between those waves, together with the fact that the (constant) Hamiltonian along either one appears as a *Lagrange multiplier* associated with the other.

Theorem 1. *If $(\rho, u) = (R, U)(x - \sigma t)$ is a traveling wave solution to* (15) *of speed σ such that R is positive, bounded and bounded away from zero, then there exist a unique $j \in \mathbb{R}$ and $(v, u) = (V, W)(y + jt)$ a traveling wave solution to* (13) *such that*

$$R(\xi)\,(U(\xi)\,-\,\sigma)\;=\;\frac{W(\zeta)\,-\,\sigma}{V(\zeta)}\;\equiv:\;j\,,\tag{21}$$

$$\zeta = Z(\xi)\,,\;Z'\;=\;R\;=\;\frac{1}{V\circ Z}\,,\;U\;=\;W\circ Z\,.\tag{22}$$

Conversely, if $(v, u) = (V, W)(y + jt)$ is a traveling wave solution to (13) *such that V is positive, bounded and bounded away from zero, then there exist a unique $\sigma \in \mathbb{R}$ and $(\rho, u) = (R, U)(x - \sigma t)$ a traveling wave solution to* (15) *satisfying* (21)-(22). *More precisely, the governing ODE for the Eulerian profile R is parametrized by some real number μ, and coincides with the Euler–Lagrange equation $\delta\mathscr{L} = 0$ for the Lagrangian*

$$\mathscr{L}\,:=\,\mathscr{E}\;-\;\frac{j^2}{2\rho}\;-\;\mu\rho\,,$$

while the governing ODE for the Lagrangian profile V is parametrized by some real number λ, and coincides with the Euler–Lagrange equation $\delta\ell = 0$ for the Lagrangian

$$\ell := e - \frac{j^2 v^2}{2}\;-\;\lambda v\,.$$

The numbers λ and μ may be viewed either as Lagrange multipliers *or as values of* first integrals*: they are indeed given by*

$$\lambda\;=\;-\mathsf{L}_\rho\mathscr{L}\,,\quad \mu\;=\;-\,\mathsf{L}_v\ell\,,$$

where $\mathsf{L}_\rho\mathscr{L}$ and $\mathsf{L}_v\ell$ stand for the (formal) Legendre transforms of $\mathscr{L}$ and ℓ evaluated at profiles,

$$\mathsf{L}_\rho\mathscr{L}\,:=\,\rho_x\,\frac{\partial\mathscr{L}}{\partial\rho_x}\;-\;\mathscr{L}\,,\quad \mathsf{L}_v\ell\;:=\;v_y\,\frac{\partial\ell}{\partial v_y}\;-\;\ell\,,$$

the former being the Hamiltonian associated with $\delta\mathscr{L} = 0$, and the latter with $\delta\ell = 0$.

Proof. On the one hand, if $(\rho, u) = (R, U)(x - \sigma t)$ is a traveling wave solution to (15) of speed σ such that R is positive and bounded away from zero then

- any primitive Z of R is an increasing diffeomorphism of $\mathbb{R}$,
- by the continuity equation in (15), $R\,(U\,-\,\sigma)$ is equal to some constant j,

and we claim that $V := 1/(R \circ Z^{-1})$, $W := U \circ Z^{-1}$ give a traveling wave solution $(v,u) = (V,W)(y+jt)$ of (13). Note that the degree of freedom in the choice of the primitive Z merely yields translation in the variable $\zeta = Z(\xi)$.

On the other hand, if $(v,u) = (V,W)(y+jt)$ is a traveling wave solution to (13) such that V is positive and bounded away from zero, then

- any solution of the scalar differential equation $Z' = 1/V(Z)$ is a (global) increasing diffeomorphism of $\mathbb{R}$,
- by the first equation in (13), $W - jV$ is equal to some constant σ,

and we claim that $R := Z'$, $U := W \circ Z$ give a traveling wave solution $(\rho,u) = (R,U)(x-\sigma t)$ of (15). Note that this time the degree of freedom in the choice of the solution Z yields translation in the variable ξ (if Z^0 and Z^1 are solutions of $Z' = 1/V(Z)$ then by the mean value theorem there exists ξ_1 such that $Z^0(\xi_1) = Z^1(0)$, and by translation invariance of this differential equation we have $Z^0(\xi+\xi_1) = Z^1(\xi)$ for all $\xi \in \mathbb{R}$).

Our claims above can be proved in a most abstract way, which we are going to describe now, or can be viewed as a consequence of the computations we shall make afterwards. This abstract proof relies on the 'pivot' system that is hidden in the passage from the Euler–Korteweg system (15) in Eulerian coordinates to (8) in *mass* Lagrangian coordinates, and reads

$$\left\{\begin{aligned} \rho_0\,\partial_s \breve{v} \;&=\; \partial_\xi \breve{u}\,,\\ \rho_0\,\partial_s \breve{u} \;&=\; \partial_\xi \breve{\Sigma}\,. \end{aligned}\right. \tag{23}$$

Indeed, we pass from (15) to (23) by solving at least locally the *flow map* problem

$$\partial_t \chi = u(t,\chi)\,,\ \chi(0,\xi) = \xi\,,$$

and by setting

$$\breve{v}(t,\xi) \;=\; 1/\rho(t,\chi(t,\xi))\,,\ \breve{u}(t,\xi) \;=\; u(t,\chi(t,\xi))\,,\ \rho_0(\xi) \;=\; \rho(0,\xi)\,,$$

while we pass from (23) to (8) merely by setting

$$(\widetilde{v},\widetilde{u})(s,y_0(\xi)) \;=\; (\breve{v},\breve{u})(s,\xi)\,,\quad \widetilde{\Sigma}(\widetilde{v},\widetilde{v}_y,\widetilde{v}_{yy}) = \breve{\Sigma}(\breve{v},\breve{v}_\xi,\breve{v}_{\xi\xi})\,,\quad \frac{\mathrm{d}y_0}{\mathrm{d}\xi} \;=\; \rho_0\,.$$

In particular, the flow map χ associated with a traveling wave $(\rho,u)(t,x) = (R,U)(x-\sigma t)$ solution of (15) is such that $\chi_\sigma(t,\xi) := \chi(t,\xi) - \sigma t$ solves the autonomous ODE problem

$$\partial_t \chi_\sigma = U(\chi_\sigma) \,-\, \sigma\,,\ \chi_\sigma(0,\xi) = \xi\,.$$

Using that $R\,(U-\sigma) \equiv j$ and introducing Y a primitive of R, we get a sort of conjugation identity

$$R(\chi(t,\xi) - \sigma t) \;=\; R(\xi) \,+\, j\,t\,. \tag{24}$$

Let us now define the solution $(\widetilde{v},\widetilde{u})$ of (8) by

$$\widetilde{v}(t,Y(\xi)) \,=\, 1/\rho(t,\chi(t,\xi))\,,\ \widetilde{u}(t,Y(\xi)) \,=\, u(t,\chi(t,\xi))\,,\quad \frac{\mathrm{d}Y}{\mathrm{d}\xi} \,=\, R\,.$$

Thanks to (24) we see that

$$(\widetilde{v},\widetilde{u})(t,Y(\xi)) \,=\, (\widetilde{v}_0,\widetilde{u}_0)(Y(\xi)\,+\,j\,t)\,,\ \text{where}\ \ \widetilde{v}_0(Y(\xi)) \,:=\, 1/R(\xi)\,,\ \widetilde{u}_0(Y(\xi)) \,:=\, U(\xi)\,.$$

This exactly means that $(\widetilde{v},\widetilde{u})$ is a traveling wave solution of (8). The argument also goes backward, that is to say from a Lagrangian traveling wave solution $(\widetilde{v},\widetilde{u})(s,y) = (V,W)(y+js)$ to an Eulerian one, provided that the ODE $Y' = 1/V(Y)$ and the implicit equation (24) have a unique solution $\chi(t,\xi)$. This is the case at least locally in (t,ξ) when V is positive, bounded and bounded away from zero (which by the way implies that the solutions of $Y' = 1/V(Y)$ are global).

Let us now examine the profile equations closer. We make in parallel the computations for (15) and (13), and postpone for a while the investigation of their relationship. The profile equations for (15) and (13) read respectively

$$R\,(U\,-\,\sigma) \equiv: j\,,\quad \mathrm{D}_\xi\left(\mathsf{E}_\rho\mathscr{E}\,+\,\frac{1}{2}\,U^2\,-\,\sigma\,U\right) \,=\, 0\,,$$

$$W - jV \equiv: \sigma\,,\quad \mathrm{D}_\zeta(\mathsf{E}_v e - jW) \,=\, 0\,,$$

from which we can eliminate the velocities and find

$$\mathrm{D}_\xi\left(\mathsf{E}_\rho\mathscr{E}\,+\,\frac{j^2}{2R^2}\right) \,=\, 0\,, \tag{25}$$

$$\mathrm{D}_\zeta(\mathsf{E}_v e - j^2 V) \,=\, 0\,. \tag{26}$$

One may observe that

$$\frac{1}{2R^2} \,=\, \mathsf{E}_\rho Q\,,\quad V \,=\, \mathsf{E}_v q\,, \tag{27}$$

where

$$Q := \frac{1}{2\rho}\,,\quad q := \frac{1}{2}v^2\,.$$

(These notations Q and q are consistent with the convention we have been using for uppercase and lowercase letters since $Q = \rho q$.) Note that the variational derivatives $\mathsf{E}_\rho\mathscr{E}$ and $\mathsf{E}_v e$ in (25) and (26) are respectively evaluated at $(V, V_\zeta, V_{\zeta\zeta})$ and $(R, R_\xi, R_{\xi\xi})$, and similarly $\mathsf{E}_\rho Q$ and $\mathsf{E}_v q$ stand for $(\mathsf{E}_\rho Q)(R)$ and $(\mathsf{E}_v q)(V)$ in (27). The equations (25), (26), (27) above thus yield μ and λ such that

$$\mathsf{E}_\rho(\mathscr{E}\,-\,j^2\,Q\,-\,\mu\,\rho) \,=\, 0\,, \tag{28}$$

$$\mathsf{E}_v(e \,-\, j^2\, q \,-\, \lambda\, v) \,=\, 0\,. \tag{29}$$

This is where the definitions of the Lagrangians $\mathscr{L}$ and ℓ come from, as we recognize their Euler–Lagrange equations $\delta\mathscr{L} = 0$ and $\delta\ell = 0$ in (28) and (29) respectively. It remains to check the equivalence between the existence of a constant μ and a positive bounded (below and above) function $R = R(\xi)$ solution of (28), and the existence of a constant λ and a positive bounded (below and above) function $V = V(\zeta)$ solution of (29) with $\zeta = Z(\xi)$, $Z' = R = 1/(V \circ Z)$. Let us first point out that (28) and (29) are second order differential equations so that by the Poincaré-Bendixson theorem their bounded solutions can only be *homoclinic*, *heteroclinic*, or *periodic* solutions. Moreover, as soon as

$$R(\xi) = 1/V(\zeta),\ \mathrm{d}\zeta \,=\, R\,\mathrm{d}\xi \tag{30}$$

with R positive, bounded and bounded by below then obviously so is V, and if R tends to $\rho_\pm$ at $\pm\infty$ then V tends to $v_\pm := 1/\rho_\pm$ at $\pm\infty$. Thus a homo/hetero-clinic Eulerian orbit will (fortunately) correspond to a homo/hetero-clinic Lagrangian orbit. The same correspondence will occur for periodic orbits. Indeed, we have by change of variables

$$\int_{\xi_0}^{\xi_1} R_\xi\,\mathrm{d}\xi \,=\, -\int_{\zeta_0}^{\zeta_1} V_\zeta\,\mathrm{d}\zeta\,,$$

(with the obvious notations $\zeta_{0,1} \,=\, Z(\xi_{0,1})$), so that R is periodic of period $\xi_1 - \xi_0$ if and only if V is periodic of period $Z(\xi_1) - Z(\xi_0)$.

We finally turn to the explicit verification of the relationship between Eulerian profile equations and Lagrangian profile equations. Let us recall from (9) that

$$\mathsf{E}_v e \,=\, \mathscr{E} \,-\, \rho\,\mathsf{E}_\rho\mathscr{E} \,-\, \rho_x\,\frac{\partial\mathscr{E}}{\partial\rho_x}\,. \tag{31}$$

To be more precise we are going to use that

$$(\mathsf{E}_v e)(V, V_\zeta, V_{\zeta\zeta}) \,=\, \mathscr{E}(R, R_\xi) \,-\, R\,(\mathsf{E}_\rho\mathscr{E})(R, R_\xi, R_{\xi\xi}) \,-\, R_\xi\,\frac{\partial\mathscr{E}}{\partial\rho_x}(R, R_\xi)$$

when V and R are related through (30). For simplicity we omit to write below the dependent variables $V, V_\zeta, V_{\zeta\zeta}$, $R, R_\xi, R_{\xi\xi}$, as long as no confusion can occur. Thanks to (31) and to the analogous, though simpler, formula

$$\mathsf{E}_v q \,=\, Q \,-\, \rho\,\mathsf{E}_\rho Q\,,$$

we have

$$\mathsf{E}_v e - j^2\,\mathsf{E}_v q \,-\, \lambda \,=\, -\,R\,(\mathsf{E}_\rho\mathscr{E} \,-\, j^2\,\mathsf{E}_\rho Q \,-\, \mu) \,+\, \mathscr{E} \,-\, j^2\,Q \,-\, \mu\,R \,-\, R_\xi\,\frac{\partial\mathscr{E}}{\partial\rho_x} \,-\, \lambda\,.$$

This equivalently reads

$$\mathsf{E}_v\ell \,=\, -\,R\,\mathsf{E}_\rho\mathscr{L} \,-\, \mathsf{L}_\rho\mathscr{L} \,-\, \lambda\,, \tag{32}$$

with
$$\ell := e - j^2 q - \lambda v\,, \quad \mathscr{L} := \mathscr{E} - j^2 Q - \mu \rho\,.$$
For the moment, λ and μ are just arbitrary parameters, and
$$\mathsf{L}_\rho \mathscr{L} = \rho_x \frac{\partial \mathscr{L}}{\partial \rho_x} - \mathscr{L}$$
is not necessarily constant: in general we have
$$\mathrm{D}_x(\mathsf{L}_\rho \mathscr{L}) = -\rho_x \mathsf{E}_\rho \mathscr{L}\,.$$
For symmetry reasons (note that $(\mathscr{E}, Q, \mu, R, \xi) \leftrightarrow (e, q, \lambda, V, \zeta)$ exchanges $\mathscr{L}$ and ℓ while leaving (30) invariant), we have

$$\mathsf{E}_\rho \mathscr{L} = -V\, \mathsf{E}_v \ell - \mathsf{L}_v \ell - \mu\,, \tag{33}$$

similarly to (32). From these two equations follows the equivalence between
$$\mathsf{E}_\rho \mathscr{L} = 0\,, \quad \mathsf{L}_\rho \mathscr{L} = -\lambda\,,$$
and
$$\mathsf{E}_v \ell = 0\,, \quad \mathsf{L}_v \ell = -\mu\,.$$
□

As seen in the course of the previous proof, planar traveling waves solutions to the Euler–Korteweg system basically pertain to three types of waves, namely,

- heteroclinic waves, which are often called *kinks*,
- homoclinic waves, which are usually called *solitons*,
- periodic waves.

The sets of these waves are generically manifolds of increasing dimensions, the lowest dimension being for kinks, and the highest for periodic waves. Indeed, if we adopt for instance the Eulerian point of view, a kink is determined up to spatial translations by its endstates $\rho_\pm$, the relative momentum j, the speed σ, and the parameters λ and μ. (Note that $(\rho_-, \rho_+, j, \sigma)$ determine unique endstates in the velocity components, namely $u_\pm = \sigma + j/\rho_\pm$.) Those six parameters are not independent and must satisfy four equations, namely those given by the profile equation $\delta \mathscr{L}$ at both ends,
$$\frac{\partial \mathscr{E}}{\partial \rho}(\rho_\pm, 0) + \frac{j^2}{2\rho_\pm^2} = \mu\,,$$
and the constraints that the endpoints $(\rho_\pm, 0)$ be on the same level curve of the Hamiltonian $H := \mathsf{L}_\rho \mathscr{L}$ at level $-\lambda$,
$$\mathscr{E}(\rho_\pm, 0) - \frac{j^2}{2\rho_\pm} - \mu\,\rho_\pm = \lambda\,.$$

In other words, the mapping

$$\rho \mapsto \mathscr{L}(\rho,0) - \lambda = \mathscr{E}(\rho,0) - \frac{j^2}{2\rho} - \mu\,\rho - \lambda$$

must have double zeroes at $\rho_\pm$. Equivalently, looking at equations from the Lagrangian point of view,

$$v \mapsto \ell(v,0) - \mu = e(v,0) - \frac{j^2 v^2}{2} - \lambda\,v - \mu$$

must have double zeroes at $v_\pm$. So generically the set of heteroclinic orbits (*i.e.* kinks up to spatial translations) is a two-dimensional surface. As regards solitons, they are determined up to spatial translations by their only endstate ρ_∞, the relative momentum j, the speed σ, and the parameters λ and μ, under the constraints that

$$\rho \mapsto \mathscr{E}(\rho,0) - \frac{j^2}{2\rho} - \mu\,\rho - \lambda$$

has a double zero at ρ_∞. These two equations being independent, the set of homoclinic orbits (solitons up to spatial translations) is thus a three-dimensional manifold. To finish with, a periodic wave is determined up to spatial translation by say its trough ρ_*, and still (j,σ,λ,μ) such that

$$\mathscr{E}(\rho_*,0) - \frac{j^2}{2\rho_*} - \mu\,\rho_* = \lambda\,.$$

Thus closed orbits (*i.e.* periodic waves up to spatial translations) generically form a four-dimensional manifold.

It is usually expected that the fewer degrees of freedom some special solution has, the stronger its stability. This is roughly what happens here, as we shall see in the next section.

Let us now make a few comments on the existence of heteroclinic/homoclinic/periodic waves when the energy is of the form in (11), or equivalently (12). In this case, the governing ODEs of those waves read, in canonical variables[4]

$$(34)\qquad \left\{\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}\xi} &= \frac{\partial H}{\partial \pi}(\rho,\pi)\,,\\ \frac{\mathrm{d}\pi}{\mathrm{d}\xi} &= -\frac{\partial H}{\partial \rho}(\rho,\pi)\,, \quad H = \frac{1}{2}\,\frac{\pi^2}{K(\rho)} - F(\rho) + \frac{j^2}{2\rho} + \mu\,\rho\,, \end{aligned}\right.$$

from the Eulerian point of view or, from the Lagrangian point of view

$$(35)\qquad \left\{\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}\zeta} &= \frac{\partial h}{\partial w}(v,w)\,,\\ \frac{\mathrm{d}w}{\mathrm{d}\zeta} &= -\frac{\partial h}{\partial v}(v,w)\,, \quad h = \frac{1}{2}\,\frac{w^2}{\kappa(v)} - f(v) + \frac{j^2 v^2}{2} + \lambda\,v\,. \end{aligned}\right.$$

[4]These variables are just (ρ,π) where π is the partial derivative of $\mathscr{L}$ with respect to ρ_x at fixed ρ. When the energy is of the form in (11), $\pi = K(\rho)\,\rho_x$.

It follows from Theorem 1 that there is a one-to-one correspondance between orbits of (34) lying on the level curve $H = -\lambda$ and orbits of (35) lying on the level curve $h = -\mu$. Otherwise, it is worth noting that by changing coordinates into $\widetilde{\pi} = \pi/\sqrt{K(\rho)}$ and $\widetilde{w} = w/\sqrt{\kappa(v)}$, the ODEs above 'simplify' into

$$\left\{ \begin{array}{l} \sqrt{K(\rho)}\,\dfrac{\mathrm{d}\rho}{\mathrm{d}\xi} \;=\; \widetilde{\pi}\,, \\ \sqrt{K(\rho)}\,\dfrac{\mathrm{d}\widetilde{\pi}}{\mathrm{d}\xi} \;=\; F'(\rho) \,+\, \dfrac{j^2}{2\rho^2} \,-\, \mu\,, \end{array} \right. \tag{36}$$

$$\left\{ \begin{array}{l} \sqrt{\kappa(v)}\,\dfrac{\mathrm{d}v}{\mathrm{d}\zeta} \;=\; \widetilde{w}\,, \\ \sqrt{\kappa(v)}\,\dfrac{\mathrm{d}\widetilde{w}}{\mathrm{d}\zeta} \;=\; f'(v) \,-\, j^2 v \,-\, \lambda\,. \end{array} \right. \tag{37}$$

By reparametrizing the orbits we see that the phase portraits of (36) and (37) are actually independent of the capillarity coefficient K, or equivalently κ, provided that it is positive and bounded away by zero. This shows that the existence of heteroclinic/homoclinic/periodic planar waves for the Euler–Korteweg system with an energy as in (2) does not depend on the expression of capillarity. From a physical point of view, this is rather satisfactory because K is not directly amenable to experiments[5]. Thus we are left with investigating the possible phase portraits of the reparametrized version of (37), which merely reads

$$\left\{ \begin{array}{l} \dot{v} \;=\; w\,, \\ \dot{w} \;=\; f'(v) \,-\, j^2 v \,-\, \lambda\,. \end{array} \right. \tag{38}$$

(For simplicity, we have omitted the tilda over the modified coordinate $\widetilde{w}$.) Let us mention that, physically, $p(v) := -f'(v)$ represents the pressure in the fluid. We may distinguish between several cases of pressure laws, namely

1. monotonically decaying and convex pressure law,
2. monotonically decaying and nonconvex pressure law,
3. nonconvex pressure law.

A well-known example exhibiting all three behaviours for various values of parameters is the *van der Waals* pressure law

$$p(v) \;=\; \frac{RT}{v-b} \;-\; \frac{a}{v^2}\,.$$

This law admits two transition temperatures

$$T_0 := \frac{81a}{256bR}\,,\; T_c := \frac{8a}{27bR}\,,$$

(note that the relative difference between the two is only of 2%), and we have that

[5] It is its macroscopic effect, *surface tension*, that can be measured.

1. for $T > T_0$, p is monotonically decaying and convex,
2. for $T_c < T < T_0$, p is monotonically decaying and admits two inflection points,
3. for $T < T_c$, p admits one local minimum, one local maximal, and two inflection points.

The latter case is a model for liquid-vapor mixtures. In the first two cases, the thermodynamical state of the fluid is supercritical and it is not possible to distinguish between a vapor phase and a liquid phase. The typical phase portrait for (38) in the first case is depicted on Figure 1. Of course, if λ is taken too large, the *Rayleigh line* of equation $p = -j^2 v - \lambda$ fails to intersect the pressure graph, in which case no fixed point and *a fortiori* no bounded traveling wave arises at all. Another simple remark that can be drawn from this picture is that for $j = 0$, the only bounded traveling 'wave' is the fixed point at which p assumes the value $-\lambda$ (if it exists). In other words, for monotone convex pressure laws the Euler–Korteweg system in Lagrangian coordinates does not admit any *stationary* soliton nor any stationary (spatially) periodic solution. Recalling that j is actually not a physical speed but a momentum density, this means that in Eulerian coordinates the Euler–Korteweg system does not admit any bounded traveling wave without mass transfer across it. Otherwise, phase portraits for monotone nonconvex and nonmonotone nonconvex are similar, see Figures 2, 3, 4 (on pp. 44-45) for three typical ones that can be obtained with for instance the van der Waals law under critical temperature T_c.

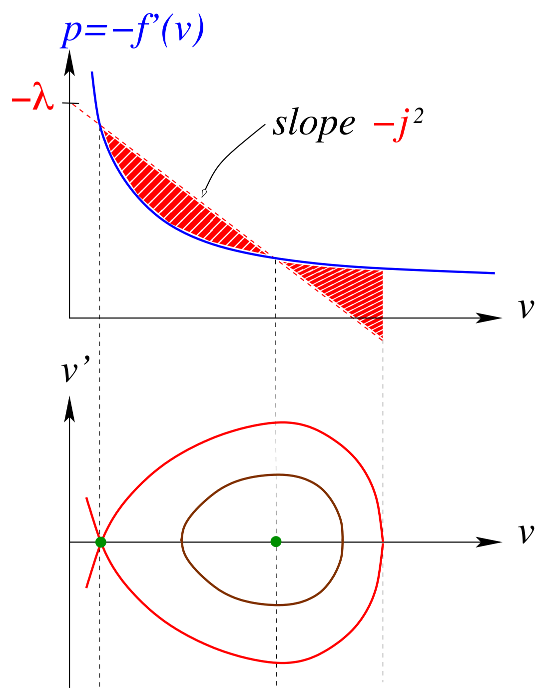

Figure 1: A monotone convex pressure law and the associated phase portrait. Dashed areas are supposed to be equal, and determine the height of the soliton.

3 Stability analysis of planar traveling waves

As general rule, stability should not depend on coordinates. This is not completely obvious when speaking of Lagrangian coordinates *versus* Eulerian coordinates though. Among other things, we shall point out why it is so on the specific waves we are considering. Another issue of interest in this section is the independence of stability upon the specific choice of capillarity (we have already seen that the *existence* of planar traveling waves does not depend on capillarity provided that it remains positive everywhere).

3.1 Tools and issues

The purpose of this section is to review various tools involved in the stability analysis of planar traveling waves, namely Boussinesq's moment of instability, Krein's signature, Evans functions, Whitham's theory of modulated equations. As far as possible, we apply them to the Euler–Korteweg system, and point out relationships between them. The actual results of stability/instability that can be drawn from those tools will be detailed in Section 3.2.

A first, natural approach when investigating the stability of a special solution to a Hamiltonian evolution equation is to try and see whether the Hamiltonian itself has any chance to play the role of a *Lyapunov* function. Of course one of the requirements for being a Lyapunov function is trivially satisfied by Hamiltonians, which are (almost) by definition constant along solutions. A more delicate thing is the behavior of the Hamiltonian in the 'neighborhood' of the special solution considered. For a Hamiltonian PDE like (4) (or similarly (14)) a candidate for a Lyapunov function is not exactly the (local) Hamiltonian $\mathscr{H}$ (or $\hbar$) but the functional $\int \mathscr{H}\,\mathrm{d}x$ (or $\int \hbar\,\mathrm{d}y$). A first difficulty is that this integral has no reason to be convergent. In several space dimensions, there is hardly any way to make an integral of this type convergent when $\mathscr{H}$ is evaluated at states that are close to a planar wave (because of obvious lack of decay in directions of the plane). However, as long as we are concerned with one-dimensional stability, there is a rather simple remedy for this lack of convergence, which consists in considering the integral

$$\int (\mathscr{H}(\rho,u,\rho_x)\,-\,\mathscr{H}(R,U,R_x))\,\mathrm{d}x$$

(or similarly $\int(\hbar(v,u,v_y)\,-\,\hbar(V,W,V_y))\,\mathrm{d}y$) to investigate the stability of a wave whose profile is (R,U) (or (V,W)). If (R,U) is a profile homoclinic to (ρ_∞,u_∞), an even simpler alternative is

$$\int (\mathscr{H}(\rho,u,\rho_x)\,-\,\mathscr{H}(\rho_\infty,u_\infty,0))\,\mathrm{d}x\,.$$

Yet this is not enough, because (R,U) has no reason to be a critical point of this integral, or equivalently to cancel its variational derivative $\delta\mathscr{H}$. This time the remedy lies in the observation that (R,U) is a critical point of a modified functional. We almost saw it in Theorem 1. It was just hidden by the fact that we (intentionally) eliminated velocities.

Indeed, the profile equations for (15) read, in vectorial form,

$$\mathrm{D}_\xi \left(\delta\mathscr{H} \,-\, \sigma \begin{pmatrix} R \\ U \end{pmatrix} \right) \,=\, \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the variational derivarive $\delta\mathscr{H}$ is (as usual) evaluated at (R,U). Observing that the second term can also be written as a variational derivative, namely the one of $\mathscr{Q} := \rho\, u$, at (R,U), we may equivalently write the system above as

$$\delta(\mathscr{H} \,-\, \sigma\,\mathscr{Q} \,-\, \mu_1\,\rho \,-\, \mu_2\,u) \,=\, 0\,, \tag{39}$$

for some real numbers μ_1 and μ_2. This equation precisely means that the profile (R,U) is a critical point of the *modified Hamiltonian* $\mathscr{H} \,-\, \sigma\,\mathscr{Q}$ under constraints on ρ and u associated respectively with μ_1 and μ_2 as Lagrange multipliers. The 'modifier' $\mathscr{Q}$ has been called an *impulse* by Benjamin [4]. Eq. (39) is to be understood as the system

$$\left\{ \begin{array}{l} \mathsf{E}_\rho(\mathscr{H} \,-\, \sigma\,\mathscr{Q}) \,=\, \mu_1\,, \\ \mathsf{E}_u(\mathscr{H} \,-\, \sigma\,\mathscr{Q}) \,=\, \mu_2\,. \end{array} \right.$$

Comparing these equations with those in the proof of Theorem 1 we see that

$$\mu_1 \,=\, \mu \,-\, \frac{1}{2}\,\sigma^2\,, \quad \mu_2 \,=\, j\,.$$

Unsurprisingly, we can make similar observations from the Lagrangian point of view. In deed, the profile equations for (13) read

$$\delta(\mathscr{h} \,-\, j\,\mathscr{p} \,-\, \lambda_1\,v \,-\, \lambda_2\,u) \,=\, 0\,, \tag{40}$$

or equivalently

$$\left\{ \begin{array}{l} \mathsf{E}_v(\mathscr{h} \,-\, j\,\mathscr{p}) \,=\, \lambda_1\,, \\ \mathsf{E}_u(\mathscr{h} \,-\, j\,\mathscr{p}) \,=\, \lambda_2\,, \end{array} \right.$$

where $\mathscr{p} := v\,u$, and in fact $\lambda_1 = \lambda \,-\, j\,\sigma$, $\lambda_2 = \sigma$. Therefore, a better candidate for a Lyapunov function is, in the one-dimensional Euler framework,

$$\int \mathscr{M}\,\mathrm{d}x\,, \quad \mathscr{M} \,:=\, [\mathscr{H} \,-\, \sigma\,\mathscr{Q} \,-\, (\mu \,-\, \tfrac{1}{2}\,\sigma^2)\,\rho \,-\, j\,u]\,,$$

and in the Lagrangian framework

$$\int \mathscr{m}\,\mathrm{d}y\,, \quad \mathscr{m} := [\mathscr{h} \,-\, j\,\mathscr{p} \,-\, (\lambda \,-\, j\,\sigma)\,v \,-\, \sigma\,u]\,,$$

where for simplicity we have used square brackets to denote the difference between the Hamiltonians evaluated at a perturbed state and those evaluated at a reference state (as seen above, we typically have $[\mathscr{H}] = \mathscr{H}(\rho,u,\rho_x) \,-\, \mathscr{H}(R,U,R_x)$, or more simply $[\mathscr{H}] = \mathscr{H}(\rho,u,\rho_x) \,-\, \mathscr{H}(\rho_\infty,u_\infty,0)$ when we consider solitary waves homoclinic to (ρ_∞,u_∞)).

With these notations, Eqs. (39) and (40) are merely the Euler–Lagrange equations $\delta\mathscr{M} = 0$ and $\delta\mathcal{m} = 0$ associated with the modified Hamiltonians $\mathscr{M}$ and $\mathcal{m}$. Remarkably enough, a straightforward calculation shows that, under the constraints $\rho\,(u-\sigma) = j$, $u - jv = \sigma$, those modified Hamiltonians merely coincide with the Lagrangians

$$\mathscr{L} \,=\, \mathscr{E} \,-\, \frac{j^2}{2\rho} \,-\, \mu\rho\,, \quad \ell \,=\, e - \frac{j^2 v^2}{2} \,-\, \lambda v$$

defined in Theorem 1. It turns out that the Eulerian functional $\int \mathscr{M}\,\mathrm{d}x$ and the Lagrangian function $\int \mathcal{m}\mathrm{d}y$ coincide, as shown below.

Lemma 1. *Assume that*

$$\rho\,(u\,-\,\sigma) \equiv j\,, \quad w - jv \equiv \sigma\,, \quad \rho(x) \,=\, \frac{1}{v(y)}\,, \quad u(x) = w(y)\,, \quad \mathrm{d}y \,=\, \rho\,\mathrm{d}x\,,$$

$$\lim_{x\to\pm\infty}\rho(x) = \rho_\infty\,, \quad \lim_{x\to\pm\infty}u(x) = u_\infty\,, \quad \lim_{y\to\pm\infty}v(y) = v_\infty\,, \lim_{y\to\pm\infty}w(y) = u_\infty\,,$$

*these limits being attained sufficiently fast (*e.g. *exponentially fast) so that derivatives tend to zero. Let us define*

$$\mu := g_\infty + \frac{j^2}{2\rho_\infty^2}\,, \quad \lambda := -p_\infty - j^2 v_\infty\,,$$

where

$$g_\infty := \frac{\partial\mathscr{E}}{\partial\rho}(\rho_\infty, 0)\,, \quad p_\infty := -\frac{\partial e}{\partial v}(v_\infty, 0)\,.$$

Then we have the equality

$$\int \mathscr{M}\,\mathrm{d}x \,=\, \int \mathcal{m}\,\mathrm{d}y\,,$$

where

$$\mathscr{M} \,=\, \mathscr{H}(\rho, u, \rho_x) - \sigma\,\rho u - (\mu - \tfrac{1}{2}\,\sigma^2)\,\rho - j\,u - \mathscr{H}(\rho_\infty, u_\infty, 0) + \sigma\,\rho_\infty u_\infty + (\mu - \tfrac{1}{2}\,\sigma^2)\,\rho_\infty + j\,u_\infty\,,$$

$$\mathcal{m} \,=\, \hbar(v, w, v_y) - j\,v\,w - (\lambda - j\,\sigma)\,v - \sigma\,w - \hbar(v_\infty, u_\infty, 0) + j\,v_\infty\,u_\infty + (\lambda - j\,\sigma)\,v_\infty + \sigma\,u_\infty\,.$$

Proof. As remarked above, we have

$$\mathscr{M} \,=\, \mathscr{L}(\rho, u, \rho_x) \,-\, \mathscr{L}(\rho_\infty, u_\infty, 0)\,, \quad \mathcal{m} \,=\, \ell(v, w, v_y) \,-\, \ell(v_\infty, u_\infty, 0)\,,$$

hence

$$\int \mathscr{M}\,\mathrm{d}x \,=\, \int \left(\mathscr{E}(\rho, \rho_x) \,-\, \mathscr{E}(\rho_\infty, 0) \,-\, \frac{j^2}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_\infty}\right) \,-\, \mu(\rho - \rho_\infty)\right) v\,\mathrm{d}y\,.$$

Recalling that

$$e(v, v_y) = v\,\mathscr{E}(\rho, \rho_x)\,,$$

which implies in particular at infinity

$$\mathscr{E}(\rho_\infty, 0) \;=\; \rho_\infty\, g_\infty \;-\; p_\infty\,,$$

and taking into account the definitions of μ and λ, we easily recognize that

$$\int \mathscr{M}\,\mathrm{d}x \;=\; \int \Big(\mathscr{e}(v, v_y) \;-\; \mathscr{e}(v_\infty, 0) \;-\; \frac{j^2}{2}\,(v^2 - v_\infty^2) \;-\; \lambda\,(v - v_\infty)\Big)\,\mathrm{d}y \;=\; \int \mathscr{m}\,\mathrm{d}y\,.$$

□

As a further remark, let us point out that the integral $\int \mathscr{M}\,\mathrm{d}x \;=\; \int \mathscr{m}\,\mathrm{d}y$ along a solitary wave is invariant by translation in x (or y) of this wave. This is just because solitary waves are critical points of the (autonomous) Hamiltonian $\mathscr{M}$ (or equivalently $\mathscr{m}$), and fully justifies the following.

Definition 1. *For a (planar) solitary wave solution of density ρ_∞ at infinity, relative momentum j, and velocity σ, we call* moment of instability of Boussinesq[6]

$$\mathsf{M}(\rho_\infty, j, \sigma) \;:=\; \int \mathscr{M}\,\mathrm{d}x \;=\; \int \mathfrak{m}\,\mathrm{d}y\,,$$

where $\mathscr{M}$ and $\mathfrak{m}$ are defined, as in Lemma 1, respectively along the Eulerian profile $(\rho, u) = (R, U)$ and the Lagrangian profile $(v, w) = (V, W)$ of the solitary wave.

Remark 1. *In the particular case when the energy is as in* (11), (12), *we see that the constrained energies reduce to*

$$\mathscr{M} \;=\; K(R)\,R_x^2\,,\ \mathfrak{m} \;=\; \kappa(V)\,V_y^2\,,$$

along the profiles, which readily shows why

$$\int \mathscr{M}\,\mathrm{d}x \;=\; \int \mathfrak{m}\,\mathrm{d}y$$

by change of variables ($R(x) = 1/V(y)$, $\mathrm{d}y = R\,\mathrm{d}x$, $\kappa(V) = R^5\,K(\rho)$). In addition, we recognize in these integrals the energy carried by the wave, a 'localized' energy usually referred to by physicists as surface tension.

At first glance, the only fact that all translated waves have the same Boussinesq moment is bad news as regards stability. There is indeed no hope that $\mathsf{M}(\rho_\infty, j, \sigma)$ have a strict minimum at any given solitary wave. A way to 'factor out' translation invariance was nevertheless pointed out in [15, 29] (see section 3.2 for more details). This projection trick turns out to be sufficient to show the *orbital stability* of heteroclinic waves (or kinks,

[6]The term "moment d'instabilité" was coined by Boussinesq himself in his monumental 1872 paper [16, p. 100], even though he could not view it as a Lyapunov function. The concept was resurrected by Benjamin [3] a century later.

see [8, Theorem 3]) because they have few degrees of freedom. The additional degree of freedom for homoclinic waves ruins the argument and something more is needed. It is in fact the convexity of $\mathsf{M}(\rho_\infty, j, \sigma)$ on the line $\{(j,\sigma)\,;\ \rho_\infty(u_\infty - \sigma) = j\}$ that determines the orbital stability (with respect to perturbations that vanish at infinity) of a given solitary wave. This was formalized by Grillakis, Shatah and Strauss [29] in an abstract setting, and used for a wide variety of Hamiltonian PDEs since then, starting with both the so-called Korteweg–de Vries equation [15] and the Boussinesq equation itself [14]. The idea behind the sufficient stability condition is that when M is strictly convex the 'unstable direction' is transverse to the level sets of Benjamin's impulse and thus that unstable direction is harmless, see §A.3 the appendix for more details. On the contrary, if at some point (ρ_∞, j, σ) we have

$$\frac{\partial^2}{\partial\sigma^2}\mathsf{M}(\rho_\infty, \rho_\infty(u_\infty - \sigma), \sigma) \leq 0\,,$$

then there exist (a curve of) nearby perturbed states along which $\int \mathscr{M}\mathrm{d}x$ is less than $\mathsf{M}(\rho_\infty, j, \sigma)$ (see for instance [8, Prop. 10], or the original proof in [15, Theorem 3.1]), which of course precludes the use of the functional $\int \mathscr{M}\mathrm{d}x$ as a Lyapunov function. This does not readily show instability of the wave, but gives a clue. Grillakis, Shatah and Strauss actually proved, in their abstract framework, that the positivity of the second derivative of the Boussinesq moment is an iff condition for orbital stability of the solitary wave. Since their result did not apply to the Korteweg–de Vries (KdV) equation (because the skew-symmetric operator of its 'natural' Hamilonian formulation is not onto), it was one purpose of [15] to prove the same result on the specific example of KdV. In fact, it was pointed out a couple of years later by Pego and Weinstein [42] that, for three types of Hamiltonian PDEs (namely, generalized KdV, Benjamin–Bona–Mahoney, and Boussinesq), the second derivative of the Boussinesq moment is related to the low frequency behavior of the so-called *Evans* function associated with the solitary wave, a consequence being that a negative second derivative of the Boussinesq moment implies *spectral instability*. This is also the case for solitary waves of the Euler–Korteweg system, see Theorem 2 below.

Speaking of unstable directions, let us leave for a while Lyapunov functions, and describe in more details the material needed for the Evans functions machinery. In order to study the stability of a traveling wave from the spectral point of view, the starting point is to make the wave stationary, which is always possible by a 'change of frame'. More precisely, for a traveling wave of speed σ in the Eulerian framework we should change the spatial variable x into $\xi := x - \sigma t$ (or $x - \sigma t n$ in several variables when the wave propagates in direction n), while for a Lagrangian traveling wave of momentum j we should change the Lagrangian mass coordinate y into $\zeta := y + jt$. Then the role of Benjamin's impulse becomes clearer because in the new coordinates the equations of motion read

$$\partial_s \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} 0 & \partial_\zeta \\ \partial_\zeta & 0 \end{pmatrix} \delta(\mathscr{h} - j\,\mathscr{p})\,, \quad \mathscr{h} = \frac{1}{2}u^2 + \mathscr{e}(v, v_\zeta)\,, \quad \mathscr{p} = v\,u\,, \tag{41}$$

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} = -\begin{pmatrix} 0 & \partial_\xi \\ \partial_\xi & 0 \end{pmatrix} \delta(\mathscr{H} - \sigma\mathscr{Q})\,, \quad \mathscr{H} = \frac{1}{2}\rho\,u^2 + \mathscr{E}(\rho, \rho_\xi)\,, \quad \mathscr{Q} = \rho\,u\,, \tag{42}$$

instead of (14) and (16). Recalling that the traveling wave equations may be written as (40) and (39), we see that by translation in y or x these waves are indeed changed to stationary solutions, say (V,W) or (R,U), of (41) or (42) respectively. We may now linearize (41) about (V,W), or (42) about (R,U). We receive the systems

$$\partial_s \begin{pmatrix} v \\ u \end{pmatrix} = \partial_\zeta J\,\mathbf{A} \begin{pmatrix} v \\ u \end{pmatrix}, \quad \mathbf{A} := \mathsf{Hess}(\mathscr{h} - j\,\mathscr{p})(V,W)\,, \tag{43}$$

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} = -\,\partial_\xi J\,\mathbf{B} \begin{pmatrix} \rho \\ u \end{pmatrix}, \quad \mathbf{B} := \mathsf{Hess}(\mathscr{H} - \sigma\mathscr{Q})(R,U)\,, \tag{44}$$

where for simplicity we have introduced the matrix

$$J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and by definition $\mathsf{Hess}\mathscr{H}(R,U)$ is the symmetric, second order, vector-valued differential operator defined by

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\int (\mathscr{H}(R+\theta\rho, U+\theta u) - \mathscr{H}(R,u))\,\mathrm{d}\xi\right)_{|\theta=0} = \int (\rho,u)\cdot \mathsf{Hess}\mathscr{H}(R,U)(\rho,u)\,\mathrm{d}\zeta$$

and similarly for other functionals.

Definition 2. *A Lagrangian traveling wave of profile (V,W) is said*

1. linearly stable *if the operator $\mathscr{A} := \partial_\zeta J\,\mathbf{A}$ is the infinitesimal generator of a strongly continuous semi-group of contractions on $H^1 \times L^2$,*

2. spectrally stable *if $\mathscr{A}$, as an unbounded operator on $H^1 \times L^2$, has no spectrum in the half-plane $\{z\,;\ \mathrm{Re} z > 0\}$.*

Similarly, an Eulerian traveling wave of profile (R,U) is said linearly stable if the operator $\mathscr{B} := -\,\partial_\xi J\,\mathbf{B}$ is the infinitesimal generator of a strongly continuous semi-group of contractions on $H^1 \times L^2$, and spectrally stable if $\mathscr{B}$ has no spectrum in the half-plane $\{z\,;\ \mathrm{Re} z > 0\}$.

As is well-known, spectral stability is necessary to have linear stability (since the existence of an unstable eigenvalue for an operator $\mathscr{A}$ prevents the semi-group generated by $\mathscr{A}$ to be contractive), but it may be not sufficient. We shall not discuss this question here (see for instance [9] for some hints regarding the Euler–Korteweg system).

Another remark is that we cannot expect *strong* spectral stability (which would mean that the spectrum of $\mathscr{A}$ entirely lies in the open half-plane $\{z\,;\ \mathrm{Re} z < 0\}$), and thus neither *asymptotic* linear stability (which would mean that the semigroup $(\mathrm{e}^{t\mathscr{A}})$ goes to zero when $t \to +\infty$) nor a fortiori asymptotic nonlinear stability. There are several reasons to that but the first one is linked to translation invariance. Indeed, the profile equation (40) is satisfied by any translated profile $(V_s, W_s)(\zeta) = (V, W)(\zeta + s)$, that is we have

$$\delta(\mathscr{h} - j\,\mathscr{p})(V_s, W_s) \,=\, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},$$

so that by differentiation with respect to s we readily get

$$\mathsf{Hess}(\mathscr{h} - j\,\mathscr{p})(V, W)(\partial_\zeta V, \partial_\zeta W) \,=\, 0\,,$$

which means that the derivative of the profile is in the kernel of $\mathbf{A}$ and thus also of $\mathscr{A}$. This is a common feature of all translation invariant waves, and it is not the worst. If we have more degrees of freedom, for instance if we consider solitary waves parametrized by j, we may also differentiate with respect to j and receive

$$\mathsf{Hess}(\mathscr{h} - j\,\mathscr{p})(V, W)(\partial_j V, \partial_j W) \,=\, \delta\mathscr{p}(V, W) \,+\, \begin{pmatrix} \partial_j \lambda_1 \\ \partial_j \lambda_2 \end{pmatrix}. \tag{45}$$

Recalling that

$$\delta\mathscr{p}(V, W) \,=\, \begin{pmatrix} W \\ V \end{pmatrix}, \quad \mathscr{A} \,=\, \partial_\zeta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathsf{Hess}(\mathscr{h} - j\,\mathscr{p})(V, W)\,,$$

we obtain from Eq. (45) that

$$\mathscr{A} \begin{pmatrix} \partial_j V \\ \partial_j W \end{pmatrix} = \begin{pmatrix} \partial_\zeta V \\ \partial_\zeta W \end{pmatrix}.$$

In other words, for solitary waves the linearized operator $\mathscr{A}$ (and also its Eulerian counterpart $\mathscr{B}$) has a *Jordan block* of size at least 2 associated with the eigenvalue zero. We could nevertheless hope for a *spectral gap* up this unavoidable Jordan chain at zero. This is not the case though. For it can be shown that, for saddle-point connections (either kinks or solitons), the whole imaginary axis is made of essential spectrum, see [8, Theorem 3.6]. More precisely, the essential spectrum of $\mathscr{A}$ (and of $\mathscr{B}$) coincides with the imaginary axis (and since $\mathscr{A}$ is vector-valued, its essential spectrum cannot be shifted merely by the usual trick of considering weighted functional spaces).

From now on, we concentrate on locating possible spectrum of $\mathscr{A}$ (or $\mathscr{B}$) in the right half-plane. As a preliminary remark we may observe that the point spectrum has two symmetries:

- since the operator $\mathscr{A}$ (or $\mathscr{B}$) is real-valued, its spectrum is invariant by complex conjugation;

- since $\mathscr{A} = \mathscr{J}\mathbf{A}$ with $\mathscr{J}$ skew-adjoint and $\mathbf{A}$ self-adjoint, its point spectrum is invariant by $\tau \mapsto -\tau$ (for the spectrum of $\mathbf{A}\mathscr{J} = -\mathscr{A}^*$ is opposite to the spectrum of $\mathscr{A}$, and if $\mathbf{A}\mathscr{J}u = \tau u$ with $\tau \neq 0$ and $u \neq 0$ then $\mathscr{J}u \neq 0$ and $\mathscr{A}\mathscr{J}u = \tau\mathscr{J}u$, thus τ is an eigenvalue of $\mathscr{A}$).

So the occurrence of an eigenvalue anywhere outside the imaginary axis implies spectral instability. Furthermore, the observation above is the starting point for defining the *Krein signature* [30, 39] of an eigenvalue τ: denoting by I_τ the real invariant space associated with the eigenvalues τ, $\overline{\tau}$, $-\tau$, $-\overline{\tau}$, the Krein signature of τ is 0 if $\mathscr{A}_{|I_\tau}$ is indefinite, $+1$ if $\mathscr{A}_{|I_\tau}$ is definite positive, and -1 if $\mathscr{A}_{|I_\tau}$ is definite negative. It turns out that the Krein signature is especially interesting for purely imaginary eigenvalues. Indeed, if τ is of nonzero real part, its Krein signature is zero,whereas the Krein signature of $\tau \in i\mathbb{R}$ is nonzero in general, with the additional bifurcation result that colliding purely imaginary eigenvalues of opposite Krein signatures generically leave the imaginary axis as a quadruplet $(\tau, \overline{\tau}, -\tau, -\overline{\tau})$ where $\tau \notin (\mathbb{R} \cup i\mathbb{R})$. This point of view has been used in recent work [22, 31, 35, 36, 37] to study the stability of various equilibria, and in particular periodic waves, in both abstract Hamiltonian PDEs and various examples of them (Schrödinger, KdV, Gross-Pitaevskii).

Let us now draw a rather striking information from Eq. (45) (or its counterpart from the Eulerian point of view). Recalling Definition 1 for the Boussinesq moment M, and using Eq. (40), we can indeed infer from Eq. (45) that the second derivative of M with respect to j at fixed $(v_\infty, u_\infty, \sigma = u_\infty - jv_\infty)$, is

$$\partial_j^2 \mathsf{M}(\rho_\infty, j, u_\infty - jv_\infty) \,=\, -\int (\partial_j V, \partial_j W) \cdot \mathbf{A} \begin{pmatrix} \partial_j V \\ \partial_j W \end{pmatrix} \mathrm{d}\zeta \,.$$

Similarly, the second derivative of M with respect to σ at fixed $(\rho_\infty, u_\infty, j = \rho_\infty(u_\infty - \sigma))$, is

$$\partial_\sigma^2 \mathsf{M}(\rho_\infty, \rho_\infty(u_\infty - \sigma), \sigma) \,=\, -\int (\partial_\sigma R, \partial_\sigma U) \cdot \mathbf{B} \begin{pmatrix} \partial_\sigma R \\ \partial_\sigma U \end{pmatrix} \mathrm{d}\xi \,,$$

see §A.3 in the appendix for more details. Therefore, when M is strictly convex on the line $\{(j, \sigma)\,;\ \rho_\infty(u_\infty - \sigma) = j\}$, the self-adjoint operator $\mathbf{A}$ (as well as $\mathbf{B}$) necessarily has spectrum in the left-half plane, which is in fact point spectrum. This may seem to be bad news again because the number of eigenvalues of $\mathbf{A}$ in the left-half plane controls, according to [42, Theorem 3.1], the number of unstable eigenvalues of $\mathscr{A}$, so that there is in principle room for (at least) one unstable eigenvalue of $\mathscr{A}$. Nevertheless and remarkably enough, the strict convexity of M serves to eliminate that possibility. Indeed, according to the Grillakis–Shatah–Strauss theory, the strict convexity of M together with the facts that $\mathbf{A}$ has a single negative eigenvalue and has its kernel exactly spanned by the derivative of the profile, even imply the *orbital stability* of the wave, see again §A.3 for a few more details.

In other words, the strict convexity of M provides a sufficient condition for stability. A somehow more delicate problem is whether its concavity implies *instability* - a problem which was dealt with by Grillakis, Shatah, and Strauss under the assumption that the Hamiltonian operator $\mathscr{J}$ be onto, which is obviously violated here[7] by the differential operator $\mathscr{J} = \partial_\zeta J$. In order to prove instability, the *Evans function* is a useful, alternative

[7] We might cope with this problem by using instead the Hamiltonian formulation in Clebsch coordinates (see §A.2), in which the Hamiltonian operator is merely a skew-symmetric matrix $\mathbf{J}$.

tool. Evans functions were coined and investigated from the topological point of view in the framework of dissipative PDEs in [1], but are also widely used in Hamiltonian frameworks (see in particular the seminal paper by Pego and Weinstein [42], or the more recent work in [19, 34]). Let us mention that they have also been used in conjunction with the Krein signature approach in [37].

The idea behind Evans functions is *spatial dynamics*, which consists in viewing the eigenvalue equations

$$\mathscr{A}\begin{pmatrix} v \\ w \end{pmatrix} = \tau \begin{pmatrix} v \\ w \end{pmatrix} \tag{46}$$

as a dynamical system in the ζ-variable. To be more concrete, let us observe that

$$\mathscr{A} \,=\, \partial_\zeta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{Hess}\mathscr{e}(V) & -j \\ -j & 1 \end{pmatrix} = \partial_\zeta \begin{pmatrix} -j & 1 \\ \mathsf{Hess}\mathscr{e} & -j \end{pmatrix},$$

$$\mathsf{Hess}\mathscr{e} \,=\, -\,\mathrm{D}_\zeta \alpha\, \mathrm{D}_\zeta \,+\, \gamma\,,$$

$$\alpha := \frac{\partial^2 \mathscr{e}}{\partial v_\zeta^2}(V, V_\zeta)\,, \quad \gamma \,:=\, \frac{\partial^2 \mathscr{e}}{\partial v^2}(V, V_\zeta) \,-\, \mathrm{D}_\zeta \left(\frac{\partial^2 \mathscr{e}}{\partial v \partial v_\zeta}(V, V_\zeta) \right),$$

and assume for what follows that α only takes positive values (note that $\alpha(\zeta) = \kappa(V(\zeta))$ in the special case (12)). Then the eigenvalue equations (46) are equivalent to the 4×4 system of ODEs

$$(\mathbb{D}(\zeta)\mathbb{V})_\zeta \,=\, \mathbb{E}(\tau)\,\mathbb{V} \tag{47}$$

with

$$\mathbb{V} := \begin{pmatrix} v \\ v_\zeta \\ v_{\zeta\zeta} \\ w \end{pmatrix}, \ \mathbb{D}(\zeta) \,:=\, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & -\alpha_\zeta & -\alpha & -j \\ -j & 0 & 0 & 1 \end{pmatrix}, \ \mathbb{E}(\tau) \,:=\, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \tau \\ \tau & 0 & 0 & 0 \end{pmatrix}. \tag{48}$$

Note that $\mathbb{D}(\zeta)$ is nonsingular since $\alpha(\zeta) \neq 0$. Furthermore, when (V, W) is either a kink or a soliton, we can show that for $\mathrm{Re}\tau > 0$ the system (47) is asymptotically hyperbolic, which means that the limit matrices

$$\mathbb{A}_\pm(\tau) \,:=\, \lim_{\zeta \to \pm\infty} \mathbb{D}(\zeta)^{-1} \mathbb{E}(\tau)\,,$$

do not have any purely imaginary spectrum. This is due to the fact that the endpoints $(v_\pm, w_\pm)$ are necessarily saddle point of the profiles ODEs, see for instance [6, Lemma 1]. Therefore a way to formulate the existence of a nontrivial solution to the eigenvalue equations (46) tending to zero at infinity is to require the vanishing of a *Wronskian* $d(\tau)$ made of solutions of (47) tending to zero at either $-\infty$ or $+\infty$, provided that the dimensions of the 'unstable subspace' (the one of solutions going to zero at $-\infty$) and of the 'stable subspace' (of solutions going to zero at $+\infty$) be complementary. In practice we find their dimensions by a perturbation/connectedness argument. Here they are both two-dimensional. When constructed carefully (following [42, 1]), d is as smooth a function of τ as $\mathscr{A}$. Since the latter depends linearly on τ, d depends analytically on τ. Still, this remains an abstract object since there is no general method for solving variable coefficients systems of ODEs, and therefore an Evans function is hardly ever known explicitly. To really be useful the Evans function d must be constructed in such a way that it has a smooth continuation to a neighborhood of $\tau = 0$, in which we are likely to infer its local behavior from properties of the underlying profile. The technique to extend Evans functions near 0 (which belongs to the essential spectrum of $\mathscr{A}$) is by now well-known, and usually referred to as the *gap lemma* [26, 34].

In particular if $(v_\pm, w_\pm) = (v_\infty, w_\infty)$ is the endstate of a soliton, the following result (of which a detailed proof is given in [6, Lemma 1] in the Eulerian framework) makes the connection between the local behavior of the Evans function and the second derivative of the Boussinesq moment (also see [19, 42, 50]).

Theorem 2. *Let us consider the operator*

$$\mathscr{A} \;=\; \partial_\zeta \mathsf{J}\mathsf{Hess}(\mathscr{h} - j\mathscr{q})(V, W)$$

where (V, W) is the profile of a traveling wave solution of (13) *that is homoclinic to (v_∞, w_∞) and whose momentum of propagation is j. Similarly, we consider*

$$\mathscr{B} \;=\; -\partial_\xi \mathsf{J}\mathsf{Hess}(\mathscr{H} - \sigma\mathscr{Q})(R, U)$$

where (R, U) is the profile of a traveling wave solution of (15) *that is homoclinic to (ρ_∞, u_∞) and whose speed of propagation is σ. We assume that these waves are related to each other as in Theorem 1, and in particular that*

$$\rho_\infty = 1/v_\infty\,, \quad \rho_\infty(u_\infty - \sigma) = j\,,$$

and we consider the Boussinesq moment M as defined in Lemma 1. Then there exist smooth functions $d : [0, \infty) \to \mathbb{R}$, $D : [0, \infty) \to \mathbb{R}$ such that

1. *For all $\tau > 0$, $d(\tau) = 0$, respectively $D(\tau) = 0$, if and only if the eigenvalue equations* (46)*, respectively those associated with $\mathscr{B}$, have a nontrivial solution;*

2. *For $\tau \gg 1$, $d(\tau) > 0$, $D(\tau) > 0$;*

3. *Both d and D have (at least) double zeroes at $\tau = 0$, and there exist positive numbers ν and υ such that*

$$d''(0) = \nu \partial_j^2 \mathsf{M}(\rho_\infty, j, u_\infty - j v_\infty)\,, \quad D''(0) = \upsilon \partial_\sigma^2 \mathsf{M}(\rho_\infty, \rho_\infty(u_\infty - \sigma), \sigma)\,.$$

Observing that

$$\beta := \frac{\partial^2}{\partial\sigma^2}\mathsf{M}(\rho_\infty, \rho_\infty(u_\infty - \sigma), \sigma) \;=\; \rho_\infty^2 \frac{\partial^2}{\partial j^2}\mathsf{M}(\rho_\infty, j, u_\infty - j v_\infty)\,,$$

we readily infer from Theorem 2 and the mean value theorem that if $\beta < 0$ then both the Lagrangian and the Eulerian solitary wave are spectrally unstable (the Evans functions d and D vanish somewhere on the positive real axis $(0, +\infty)$).

Corollary 1. *Under the assumptions of Theorem 2, if the moment of instability* $\mathsf{M}(\rho_\infty, j, \sigma)$ *is strictly concave on the line* $\{(j,\sigma)\,;\ \rho_\infty(u_\infty - \sigma) = j\}$*, then the solitary wave is spectrally unstable in both the Eulerian and the Lagrangian frameworks.*

Let us now turn to Evans functions (and other tools) for periodic waves. As noted by Gardner [27], Evans functions are easier to construct for periodic waves than for saddle-points connecting orbits. However, if we are to consider arbitrary perturbations of the wave (and not only 'co-periodic' perturbations), the Evans function will consist of a Wronskian depending not only on the complex number τ but also on a so-called *Floquet multiplier*[8], say $\gamma \in S^1$ (a unitary complex number). Indeed, for a periodic profile (V, W) the spectrum of $\mathscr{A}$ in L^∞ consists entirely of *continuous spectrum* (which means that there are no isolated eigenvalues at all, see [23, p. 1487]), and more precisely it is the set of what Gardner called γ-eigenvalues [25]. The definition of γ-eigenvalues comes out naturally once the eigenvalue equations (46) are rewritten as a system of ODEs like (47), for which the existence of a nontrivial bounded solution is equivalent to the existence of an eigenvalue γ on the unit circle of the *monodromy matrix* $\mathbb{S}(Z;\tau)$, in which Z is the wavelength (or profile period), and $\mathbb{S}(\zeta;\tau)$ denotes the *fundamental solution*[9] of (47). Therefore, it suffices to define the Evans function by

$$d(\tau, \gamma) \;=\; \det(\mathbb{S}(Z;\tau) - \gamma \mathrm{Id})\,,$$

so that the γ-eigenvalues of $\mathscr{A}$ are the zeroes of $d(\cdot, \gamma)$. This apparently simple definition does not mean that it is easy to locate those zeroes. (In addition, there are subtle issues regarding multiplicities for which we refer to [25].) We may however obtain information on the zeros of $d(\cdot, \gamma)$ by perturbation arguments. In this respect, let us mention two results. The first one is due to Gardner [27] and shows that a necessary condition for the spectral stability of large wave-length periodic solutions is the *stability of the limiting soliton* (indeed, as should be clear from phase portraits, periodic orbits tend to an homoclinic orbit in the phase plane when their wavelength goes to infinity). The second result is by Serre [46] (also see the earlier work [40, 41] by Oh and Zumbrun) and shows in the framework of dissipative conservation laws that the local behavior of the Evans function near frequency $\tau = 0$ and Floquet multiplier $\gamma = 1$ is linked to the *Whitham modulated equations*. As a consequence, it gives a(nother) necessary condition for the spectral stability of large wave-length periodic solutions, which is the *hyperbolicity*[10] *of modulated equations*. We shall come back to these equations below, and see how Serre's result adapts to our (dispersive)

[8] This additional parameter somehow plays a similar role as the transverse wave vector η in multi-D Evans functions, of which we briefly speak at the end of §3.2.

[9] By definition, the column vectors of $\mathbb{S}(\zeta;\tau)$ are independent solutions of (47), and $\mathbb{S}(0;\tau) = \mathrm{Id}$.

[10] In the most general result, this hyperbolicity condition is to be understood in a weak sense (characteristics are real). Under an additional, generic condition, it is the usual hyperbolicity condition (characteristics are real *and* semi simple).

framework (also see the series of work by Bronski, Johnson, and Zumbrun [33] regarding the KdV equation). Let us complete this sketchy description by commenting on functional spaces. The γ-eigenvalues have been introduced as point spectrum of $\mathscr{A}$ viewed as an unbounded operator on L^∞. A natural question is the relationship between this L^∞ spectrum and the L^2 spectrum, because we may well imagine that a wave be unstable with respect to L^∞ perturbations, and nevertheless stable with respect to 'localized', L^2 perturbations. In this respect, let us assume that some $\tau \in \mathbb{C}$ is a γ-eigenvalue *for all* $\gamma = \mathrm{e}^{2i\pi\theta} \in S^1$. This means that there exists a smooth, bounded solution $\mathbb{V} = \mathbb{V}(\zeta;\theta)$ of (47) such that

$$\mathbb{V}(\zeta + mZ;\theta) \,=\, \mathrm{e}^{2i\pi\theta m}\mathbb{V}(\zeta;\theta)\,, \forall \zeta \in \mathbb{R},\ \forall m \in \mathbb{Z}\,. \tag{49}$$

We can synthesize $\mathbb{V}$ and define

$$\check{\mathbb{V}}(\zeta + mZ) \,:=\, \int_0^1 \mathbb{V}(\zeta;\theta)\,\mathrm{e}^{2i\pi m\theta}\,\mathrm{d}\theta\,, \quad \forall \zeta \in (0,Z)\,,\ \forall m \in \mathbb{Z}\,. \tag{50}$$

Indeed, let us recall the following basic things about *Bloch transforms*. If w is any L^1 function then the series $\sum_{n\in\mathbb{Z}} w(\zeta + nZ)$ converges in $L^1(0,Z)$, and we can define $\widehat{w} \in \mathscr{C}(\mathbb{R}/\mathbb{Z}; L^1(0,Z))$ by

$$\widehat{w}(\theta,\zeta) \,=\, \sum_{n\in\mathbb{Z}} \mathrm{e}^{-2i\pi n\theta}\, w(\zeta + nZ)\,, \forall \theta \in \mathbb{R}/\mathbb{Z}\,, \text{ for almost all } \zeta \in (0,Z)\,.$$

At fixed ζ, we may view $\widehat{w}(\cdot,\zeta)$ as the sum of a Fourier series, hence the inverse formula (which can also be obtained directly by Fubini's theorem)

$$w(\zeta + mZ) \,=\, \int_0^1 \widehat{w}(\theta,\zeta)\,\mathrm{e}^{2i\pi m\theta}\,\mathrm{d}\theta\,, \quad \forall \zeta \in (0,Z)\,,\ \forall m \in \mathbb{Z}\,.$$

This motivates (50) in the sense that $\widehat{\check{\mathbb{V}}}(\theta,\zeta) \,=\, \mathbb{V}(\zeta;\theta)$. Furthermore, if for instance $w \in \mathscr{D}(\mathbb{R})$, we see by an elementary computation that

$$\|w\|_{L^2(\mathbb{R})} \,=\, \|\widehat{w}\|_{L^2(\mathbb{R}/\mathbb{Z}\times(0,Z))}\,,$$

and by a density argument it can be shown that the *Bloch transformation* $\mathsf{B} : w \mapsto \widehat{w}$ defines an isometry from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}/\mathbb{Z}\times(0,Z))$. Therefore, our synthesized solution $\check{\mathbb{V}}$ of (47) is square integrable on $\mathbb{R}$. Finally, its first and last components yield a square integrable solution of the eigenvalue equations (46) of $\mathscr{A}$ since this differential operator has Z-periodic coefficients and thus commutes with the Bloch transformation B. In other words, if $\tau \in \mathbb{C}$ is a γ-eigenvalue *for all* $\gamma \in S^1$ then it is an eigenvalue on L^2. The argument works as well in the other way round: by Bloch transforming the eigenvalue equations we are led to find a solution to (47)(49) for all $\theta \in \mathbb{R}/\mathbb{Z}$, which exists if and only if $d(\tau,\gamma) = 0$ for all $\gamma \in S^1$. To summarize, a complex number τ is an L^2 eigenvalue of $\mathscr{A}$ if and only if the Riemann surface $\{(\tau,\gamma) \in \mathbb{C}^2\,;\ d(\tau,\gamma) = 0\}$ contains $\{\tau\}\times S^1$. After these general observations, let us point out that the stability with respect to perturbations of the same period as the wave is encoded by the restriction of the Evans function d to $\gamma = 1$ ($\theta = 0$), and more precisely is characterized by $\{\tau\,;\ \mathrm{Re}\tau > 0\,,\ d(\tau, 1) = 0\} = \emptyset$.

To finish with this tool-oriented section, let us exemplify Whitham's modulated equations. For the system (13), they were derived by Gavrilyuk and Serre in [28] by a direct asymptotic approach (instead of the variational approach advocated by Whitham himself [48, Chap. 14]). Since this is a nice computation from a not so-well known paper, we feel useful to reproduce it here (in the light of other observations we have made). First of all, let us point out that both the Hamiltonian $\mathscr{h} = \frac{1}{2}u^2 + \mathscr{e}$ and the impulse $\mathscr{p} = vu$ are associated with *local conservation laws*, namely

$$\partial_s \mathscr{h} - \partial_y \left(u\,\delta\mathscr{e} + u_y\,\frac{\partial \mathscr{e}}{\partial v_y} \right) = 0\,, \tag{51}$$

$$\partial_s \mathscr{p} + \partial_y \left(\mathscr{e} - v\,\delta\mathscr{e} - v_y\,\frac{\partial \mathscr{e}}{\partial v_y} - \frac{1}{2}u^2 \right) = 0\,, \tag{52}$$

which are easily checked to be satisfied by smooth solutions of (13). The aim of modulation theory is to find solutions having an asymptotic expansion of the form

$$(v, u)(s, y) = (v_0, u_0)(\varepsilon s, \varepsilon y, \phi(\varepsilon s, \varepsilon y)/\varepsilon) + \varepsilon\,(v_1, u_1)(\varepsilon s, \varepsilon y, \phi(\varepsilon s, \varepsilon y)/\varepsilon, \varepsilon) + o(\varepsilon)\,,$$

where the 'profiles' (v_0, u_0) and (v_1, u_1) are, say 2π, periodic in their third variable θ. Denoting by S and Y their first and second variables (here, the rescaled time and Lagrangian mass coordinate respectively), we introduce the further notations

$$k := \phi_Y\,,\ \omega := \phi_S\,,\ j := \frac{\omega}{k}\,.$$

What we expect is that the leading profile (v_0, u_0) be a 'slowly modulated' periodic wave, supposedly close in a $O(1/\varepsilon)$ domain in the (s, y) plane to a reference periodic wave $(V, W) = (V, W)(y + j_0\,s)$, $j_0 = \omega_0/k_0$. Since the periodic waves form a four-dimensional manifold (see p. 14), it is natural to seek a 4×4 'homogenized' system of equations for the evolution in the 'slow variables' (S, Y) of that modulated wave. This was done in [28]. The resulting homogenized system resembles the full Euler equations in Lagrangian coordinates (conservation laws for the mean specific volume, the mean velocity, the mean energy) supplemented with a conservation law for the mean impulse. This is not surprising *per se*, but expressing those conservation laws in a closed and exploitable form requires some algebra as well as a good choice of dependent variables.

Theorem 3 (Gavrilyuk–Serre)**.** *Assume that (V, W) is a periodic traveling wave solution of y-period $2\pi/k_0$. Then in a neighborhood of $(\langle V\rangle, k_0, \langle VW\rangle - \langle V\rangle\,\langle W\rangle)$, where the brackets $\langle\cdot,\cdot\rangle$ stand for mean values over $[0, 2\pi/k_0]$, there exists a function $\mathrm{e} = \mathrm{e}(\mathrm{v}, k, \Delta)$ such that the modulated equations for* (13) *about (V, W) read*

$$\left\{\begin{array}{l} \partial_S \mathrm{v} - \partial_Y \mathrm{u} = 0\,, \\ \partial_S \mathrm{u} + \partial_Y \mathrm{p} = 0\,, \\ \partial_S(\frac{1}{2}\mathrm{u}^2 + \mathrm{e}) + \partial_Y(\mathrm{pu} - j^2\Delta - j\,k\,\Theta) = 0\,, \\ \partial_S(\mathrm{vu} + \Delta) + \partial_Y(\mathrm{e} + \mathrm{v}\,\mathrm{p} - k\,\Theta - \frac{1}{2}\mathrm{u}^2 - 2\,j\,\Delta) = 0\,, \end{array}\right. \tag{53}$$

with the generalized Gibbs relation

$$\mathrm{de} \,=\, -\,\mathrm{p}\mathrm{dv} \,+\, \Theta\,\mathrm{d}k \,+\, j\,\mathrm{d}\Delta\,. \tag{54}$$

In addition, if e *is a* convex function *of* $(\mathrm{v}, k, \Delta/k)$*, then* (53) *is hyperbolic.*

Proof. Plugging the asymptotic expansion in (13), using that $\partial_s = \varepsilon\partial_S + \omega\,\partial_\theta$ and $\partial_y = \varepsilon\partial_Y + k\,\partial_\theta$, and retaining only the leading order terms we get that

$$\partial_\theta u_0 \,-\, j\,\partial_\theta v_0 \,=\, 0\,,\quad j\,\partial_\theta u_0 \,+\, \partial_\theta p_0 \,=\, 0\,,$$

where

$$p_0 := -\,\frac{\partial \mathscr{e}}{\partial v}(v_0, k\partial_\theta v_0) \,+\, k\,\mathrm{D}_\theta\left(\frac{\partial \mathscr{e}}{\partial v_y}(v_0, k\partial_\theta v_0)\right)\,,$$

hence

$$u_0 - j\,v_0 \,=\, \mathrm{u} \,-\, j\,\mathrm{v}\,,\quad j\,u_0 \,+\, p_0 \,=\, j\,\mathrm{u} \,+\, \mathrm{p}\,,\quad \mathrm{v} := \langle v_0\rangle\,,\ \mathrm{u} := \langle u_0\rangle\,,\ \mathrm{p} := \langle p_0\rangle\,,$$

the brackets $\langle\cdot,\cdot\rangle$ standing for mean values over the period 2π in θ. Now, if we retain the $O(\varepsilon)$ term when plugging the asymptotic expansion in (13)-(51)-(52), we receive after averaging the four equations

$$\left\{\begin{array}{l} \partial_S \mathrm{v} \,-\, \partial_Y \mathrm{u} \,=\, 0\,,\\ \partial_S \mathrm{u} \,+\, \partial_Y \mathrm{p} \,=\, 0\,,\\ \partial_S\langle \tfrac{1}{2}u_0^2 + \mathscr{e}_0\rangle \,+\, \partial_Y\left\langle p_0\,u_0 \,-\, \frac{\partial \mathscr{e}}{\partial v_y}(v_0, k\partial_\theta v_0)\,k\,\partial_\theta u_0\right\rangle \,=\, 0\,,\\ \partial_S\langle v_0 u_0\rangle \,+\, \partial_Y\left\langle \mathscr{e}_0 \,+\, v_0\,p_0 \,-\, \frac{\partial \mathscr{e}}{\partial v_y}(v_0, k\partial_\theta v_0)\,k\,\partial_\theta v_0 \,-\, \tfrac{1}{2}u_0^2\right\rangle \,=\, 0\,, \end{array}\right. \tag{55}$$

where $\mathscr{e}_0 := \mathscr{e}(v_0, k\partial_\theta v_0)$. Then, defining

$$\Theta := \left\langle \frac{\partial \mathscr{e}}{\partial v_y}(v_0, k\partial_\theta v_0)\,\partial_\theta v_0\right\rangle\,,$$

$$\mathrm{e} \,:=\, \langle \tfrac{1}{2}u_0^2 + \mathscr{e}_0\rangle \,-\, \tfrac{1}{2}\,\mathrm{u}^2\,,\quad \Delta \,:=\, \langle v_0 u_0\rangle \,-\, \mathrm{v}\,\mathrm{u}\,,$$

and differentiating these relations we see that

$$\mathrm{de} \,=\, \langle u_0\,\mathrm{d}u_0 \,-\, p_0\,\mathrm{d}v_0\,\rangle \,+\, \Theta\,\mathrm{d}k \,-\, \mathrm{u}\,\mathrm{du}\,,$$

$$\mathrm{d}\Delta \,=\, \langle v_0\,\mathrm{d}u_0 \,+\, u_0\,\mathrm{d}v_0\,\rangle \,-\, \mathrm{v}\,\mathrm{du} \,-\, \mathrm{u}\,\mathrm{dv}$$

and thus

$$\mathrm{de} \,-\, j\,\mathrm{d}\Delta \,=\, \langle (u_0 \,-\, j\,v_0)\,\mathrm{d}u_0 \,-\, (p_0 \,+\, j\,u_0)\,\mathrm{d}v_0\,\rangle \,+\, \Theta\,\mathrm{d}k \,-\, \mathrm{u}\,\mathrm{du} \,+\, j\,\mathrm{v}\,\mathrm{du} \,+\, j\,\mathrm{u}\,\mathrm{dv}.$$

Using the identities

$$\mathrm{p} \,-\, p_0 \,=\, j\,(u_0 \,-\, \mathrm{u}) \,=\, j^2\,(v_0 \,-\, \mathrm{v})\,, \tag{56}$$

this gives (54). It remains to express the last two equations in (55) in terms of $(\mathrm{v}, \mathrm{u}, k, \Delta)$. For the moment they just read

$$\begin{cases} \partial_S(\tfrac{1}{2}\mathrm{u}^2 + \mathrm{e}) + \partial_Y \langle p_0\, u_0 \rangle - \partial_Y(j\, k\Theta) = 0\,, \\ \partial_S(\mathrm{vu} + \Delta) + \partial_Y \left\langle \mathscr{e}_0 + v_0\, p_0 - \tfrac{1}{2}u_0^2 \right\rangle - \partial_Y(k\Theta) = 0\,. \end{cases}$$

To obtain the final form of (53) we use again (56) and show that

$$\langle p_0\, u_0 \rangle = \mathrm{p\,v} - j^2\,\Delta\,,$$

$$\left\langle \mathscr{e}_0 + v_0\, p_0 - \tfrac{1}{2}u_0^2 \right\rangle = \mathrm{e} + \mathrm{p\,v} - \tfrac{1}{2}\mathrm{u}^2 - 2\,j\,\Delta\,.$$

In order to have an hyperbolicity criterion for (53), Gavrilyuk and Serre pointed out that it admits the 'symmetric' equivalent form (as far as smooth solutions are concerned)

$$\begin{cases} \partial_S \mathrm{v} - \partial_Y \mathrm{u} = 0\,, \\ \partial_S \mathrm{u} - \partial_Y \left(\dfrac{\partial \widetilde{\mathrm{e}}}{\partial \mathrm{v}} \right) = 0\,, \\ \partial_S k - \partial_Y \left(\dfrac{\partial \widetilde{\mathrm{e}}}{\partial \delta} \right) = 0\,, \\ \partial_S \delta - \partial_Y \left(\dfrac{\partial \widetilde{\mathrm{e}}}{\partial k} \right) = 0\,, \quad \widetilde{\mathrm{e}}(\mathrm{v}, k, \delta) := \mathrm{e}(\mathrm{v}, k, k\delta) \end{cases} \tag{57}$$

The third equation above is just a reformulation of the natural constraint

$$\partial_S k - \partial_Y \omega = 0\,, \quad \omega = j\,k\,,$$

ensuring that the phase ϕ can be reconstructed from k and ω, and the fourth one is the reformulation of

$$\partial_S(\Delta/k) - \partial_Y(\Theta + j\,\Delta/k) = 0\,,$$

an additional conservation law for (53) that can be checked using (54). If $\widetilde{\mathrm{e}}$ is convex then (57) can be symmetrized (by Godunov's approach) thanks to the Hessian of $\widetilde{\mathrm{e}} + \mathrm{u}^2/2$ and thus is hyperbolic. □

3.2 State of the art

3.2.1 Kinks

As was pointed out by Grillakis, Shatah, and Strauss (see Remark p. 188 in [29]) "the kinks are always stable". At the spectral level, this is due to the monotonicity of the profile V, which implies by Sturm–Liouville theory that the operator $\mathsf{Hess}\mathscr{e} - j^2$ - whose kernel contains V_ζ - is monotone, and *in fine* that the linearized operator

$$\mathscr{A} = \partial_\zeta \begin{pmatrix} -j & 1 \\ \mathsf{Hess}\mathscr{e} & -j \end{pmatrix}$$

has no unstable spectrum. Similarly, $\mathsf{Hess}\mathscr{E} - j^2/R^3$ is monotone since its kernel contains R_ξ, and as a consequence

$$\mathscr{B} = -\partial_\xi \begin{pmatrix} j/R & R \\ \mathrm{Hess}\mathscr{E}(R) & j/R \end{pmatrix}$$

has no unstable spectrum (this was noticed in [5] without referring to [29]). Even though we can think of kinks as being stable 'by nature', nonlinear stability results demand a certain knowledge on the Cauchy problem. At present day, its well-posedness is only known to hold true locally in time, in Sobolev spaces of 'high' index. More precisely, let us recall the following, in which the reference, global smooth solutions can for instance be constants, or travelling wave solutions.

Theorem 4 ([10])**.** *If* $(\underline{v}, \underline{u})$ *is a global smooth solution of* (13)-(12)*, then the Cauchy problem for* (13)-(12) *is locally well-posed in* $(\underline{v}, \underline{u}) + (H^{k+1} \times H^k)$*,* $k \geq 2$*.*

Theorem 5 ([7])**.** *If* $(\varrho, \mathbf{u})$ *is a global smooth solution of* (1)-(2)*, then the Cauchy problem for* (1)-(2) *is locally well-posed in* $(\varrho, \mathbf{u}) + (H^{s+1} \times H^s)$*,* $s > d/2 + 1$*.*

Even in one space dimension, nothing is known for the Cauchy problem in the 'energy space' directed by $H^1 \times L^2$. This is why the orbital stability results are rather weak in that they do not yield global existence of perturbed solutions. We just have the following.

Theorem 6 ([8])**.** *Let* $\underline{\mathbf{U}}$ *be a global smooth solution of* (13)-(12) *(resp.* (15)-(11)*)*

$$\forall \varepsilon > 0\,,\ \exists \eta > 0\,;\ \forall \textit{ solution } \mathbf{U} \in \underline{\mathbf{U}} + \mathcal{C}([0, T[; H^3 \times H^2)\,,$$

of (13)-(12) *(resp.* (15)-(11)*)*

$$\max\Big(\|\mathbf{U}(0) - \underline{\mathbf{U}}\|_{H^1 \times L^2}\,,\ \|\mathbf{U}(0) - \underline{\mathbf{U}}\|_{L^1 \times L^1} \Big) < \eta \quad \Rightarrow$$

$$\forall t \in [0, T[\,,\ \inf_{s \in \mathbb{R}} \|\mathbf{U}(t) - \underline{\mathbf{U}}_s\|_{H^1 \times L^2} < \varepsilon\,.$$

In the case of (13)-(12) with $\kappa \equiv 1$ considered in [14], it is possible to induce a genuine stability result, with global existence: Bona and Sachs cope with the lack of control of derivatives in the energy space by differentiating the equations, which works because the principal part of the equations is linear with constant coefficient.

3.2.2 Solitons

As explained before, the stability of solitons is governed by the convexity of the moment of Boussinesq. Analytical verification of this convexity condition is far from being trivial. A fairly general result is the following.

Theorem 7 ([32])**.** *All solitary wave solutions of* (13)-(12) *with* κ *constant and* $f^{(3)}(v) < 0$*,* $f^{(4)}(v) \geq 0$*, are orbitally stable.*

Otherwise, for nonconvex pressure laws, there is numerical evidence that some solitons are stable and some others are unstable, see [8], as well as the more recent work [20] on (NSL).

3.2.3 Periodic waves

Regarding perturbations of the same period as the wave, it is possible to adapt the approach of Grillakis, Shatah and Strauss. This was done on cubic (NLS) (which corresponds to (15)-(11) with $4\rho K \equiv 1$, $F'(\rho) = \rho$) *via* the Madelung transform) by Gallay and Hărăgus [24] (also see [22] on KdV). For more general perturbations, showing a stability result is very difficult. For instance in [24], the authors obtain the *spectral stability* of *weak amplitude* periodic waves to the price of much technical effort. We have more convenient tools to prove instability. Recall from [27] that periodic waves of large wave length whose limiting soliton is unstable are themselves unstable. Another criterion is given by the relationship with Whitham's modulated equations. In a forthcoming work [11], we show the following.

Theorem 8. *Whitham's modulated equations for the Euler–Korteweg system read*

- *in the Eulerian framework*

$$(58)\quad \begin{cases} \partial_T K + \partial_X(\sigma K) = 0\,, \\ \partial_T\langle\rho_0\rangle + \partial_X\langle\rho_0 u_0\rangle = 0\,, \\ \partial_T\langle u_0\rangle + \partial_X\langle\tfrac{1}{2}u_0^2\rangle + \partial_X\langle g_0\rangle = 0\,, \\ \partial_T\langle\rho_0\, u_0\rangle + \partial_X\left\langle \rho_0\, u_0^2 + \rho_0\, g_0 + K\,(\partial_\theta\rho_0)\,\dfrac{\partial\mathscr{E}}{\partial\rho_x}(\rho_0, K\partial_\theta\rho_0) - \mathscr{E}_0 \right\rangle = 0\,, \end{cases}$$

which is endowed with the additional conservation law

(59)
$$\partial_T\langle\tfrac{1}{2}\,\rho_0\,u_0^2 + \mathscr{E}_0\rangle + \partial_X\left\langle \tfrac{1}{2}\,\rho_0\,u_0^3 + \rho_0\,u_0\,g_0 + K\,\partial_\theta(\rho_0\,u_0)\,\frac{\partial\mathscr{E}}{\partial\rho_x}(\rho_0, K\partial_\theta\rho_0)\right\rangle = 0\,,$$

- *in the Lagrangian framework*

$$(60)\quad \begin{cases} \partial_S k - \partial_Y(jk) = 0\,, \\ \partial_S\langle v_0\rangle - \partial_Y\langle w_0\rangle = 0\,, \\ \partial_S\langle w_0\rangle + \partial_Y\langle p_0\rangle = 0\,, \\ \partial_S\langle v_0\, w_0\rangle + \partial_Y\left\langle -\tfrac{1}{2}\,w_0^2 + v_0\,p_0 + e_0 - k\,(\partial_\theta v_0)\,\dfrac{\partial e}{\partial v_y}(v_0, k\partial_\theta v_0)\right\rangle = 0\,, \end{cases}$$

which is endowed with the additional conservation law

$$(61)\qquad \partial_S\langle\tfrac{1}{2}\,w_0^2 + e_0\rangle + \partial_Y\left\langle w_0\,p_0 - k\,(\partial_\theta w_0)\,\frac{\partial e}{\partial v_y}(v_0, k\partial_\theta v_0)\right\rangle = 0\,.$$

Furthermore, (60) *is equivalent to* (58) *through the relations*

$$(62)\qquad \langle\rho_0\rangle = \frac{K}{k}\,,\ \langle v_0\rangle = \frac{k}{K}\,,\quad \langle v_0\rangle = \frac{1}{\langle\rho_0\rangle}\,,\quad \langle w_0\rangle = \frac{\langle\rho_0 u_0\rangle}{\langle\rho_0\rangle}\,,$$

and

$$\mathrm{d}Y = \langle\rho_0\rangle\mathrm{d}X - \langle\rho_0 u_0\rangle\mathrm{d}T\,,\quad S = T\,.$$

In addition, the strict convexity of

$$\mathrm{e} := \langle e_0 \rangle \,+\, \tfrac{1}{2}\,\langle w_0^2 \rangle \,-\, \tfrac{1}{2}\,\langle w_0 \rangle^2 \,,$$

as a function of $(\langle v_0 \rangle, k, (\langle v_0 w_0 \rangle \,-\, \langle v_0 \rangle \, \langle w_0 \rangle)/k)$ *is equivalent to the convexity of*

$$\langle \rho_0 \rangle \, \mathrm{e} \,=\, \langle \mathscr{E}_0 \rangle \,+\, \frac{1}{2} \langle \rho_0 u_0^2 \rangle \,-\, \frac{1}{2} \frac{\langle \rho_0 u_0 \rangle^2}{\langle \rho_0 \rangle}$$

as a function of $(\langle \rho_0 \rangle, K, (\langle \rho_0 \rangle \langle u_0 \rangle - \langle \rho_0 u_0 \rangle)/K)$*. Strict convexity of these functions imply the hyperbolicity of* (58) *and* (60)*. Finally, the hyperbolicity of* (58)/(60) *in the neighborhood of a periodic traveling wave solution to* (1)/(13) *is a necessary condition for the stability of this wave.*

To finish with periodic waves, let us mention the following instability result, proved by Serre in 1994 in an unpublished work.

Theorem 9 (Serre)**.** *The* stationary *(that is, with* $j = 0$*) periodic solutions of* (13) *are unstable.*

Proof. It relies on a careful spectral analysis of $\mathsf{Hess}\, e - j^2$. For the moment we keep j arbitrary on purpose, even though j will be taken equal to zero *in fine.* For each $\theta \in \mathbb{R}/\mathbb{Z}$, the Sturm–Liouville operator $\mathsf{Hess}\, e - j^2$ has discrete spectrum on

$$L^2_\theta \,:=\, \{ \mathbb{V} \in L^2_{\mathrm{loc}}(\mathbb{R}) \,;\; \mathbb{V}(\zeta + Z; \theta) \,=\, \mathrm{e}^{2i\pi\theta} \mathbb{V}(\zeta; \theta) \,, \forall \zeta \in \mathbb{R} \} \,,$$

say

$$\lambda_0(\theta) \,\leq\, \lambda_1(\theta) \,\leq \cdots \to +\infty \,.$$

Furthermore, λ_{2k} is increasing with θ on $[0,1]$, whereas λ_{2k+1} is decreasing, see [43, pp. 293–294].Since $\partial_\zeta V$ belongs to the kernel of $\mathsf{Hess}\, e - j^2$ in L^2_0, 0 is necessarily $\lambda_1(0)$. As a consequence, we have

$$\lambda_0(\theta) \leq \lambda_1(\theta) < 0 \,, \quad \forall \theta \in]0,1] \,.$$

Denoting by P_θ the eigenspace associated with $\lambda_0(\theta)$ and $\lambda_1(\theta)$, we thus have

$$\langle \mathbb{V}, (\mathsf{Hess}\, e - j^2) \mathbb{V} \rangle \,\leq\, \lambda_1(\theta) \, \|\mathbb{V}\|^2 \,, \quad \forall \mathbb{V} \in P_\theta \,,\; \theta \in]0,1] \,.$$

Therefore, there exists $\mathbb{V}$ or zero mean value such that

$$\langle \mathbb{V}, (\mathsf{Hess}\, e - j^2) \mathbb{V} \rangle \,<\, 0 \,.$$

Defining ϕ as a primitive of $\mathbb{V}$, we have $\phi \in L^2_0$, and after integrating by parts,

$$\langle \, \partial_\zeta (\mathsf{Hess}\, e - j^2) \partial_\zeta \phi \,,\, \phi \, \rangle > 0 \,.$$

This implies that the fourth-order operator $\partial_\zeta (\mathsf{Hess}\, e - j^2) \partial_\zeta$ has at least one positive eigenvalue, say τ_0^2 for some positive τ_0. Let us denote by ϕ_0 an associated eigenvector.

Now, eliminating the velocity from the eigenvalue equations, we get the following equivalence

$$\mathscr{A}\begin{pmatrix} v \\ w \end{pmatrix} = \tau \begin{pmatrix} v \\ w \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{l} w = (\tau + j\partial_\zeta)z\,,\ \partial_\zeta z = v\,, \\ \partial_\zeta((\mathsf{Hess}\mathscr{e})(\partial_\zeta z)) \,=\, (\tau + j\partial_\zeta)^2 z\,. \end{array} \right.$$

Therefore, in the special case $j = 0$, τ_0 is an eigenvalue of $\mathscr{A}$ associated with the eigenvector $(\partial_\zeta \phi_0, \tau_0 \partial_\zeta^2 \phi_0)$. □

Of course there is no contradiction with Gallay–Hărăgus' stability result, because there are simply no stationary periodic solutions of (13)-(11) with $f(v) = 1/(2v)$ (or equivalently, $p(v) = 1/(2v^2)$, whose graph is intersected only once by any horizontal Rayleigh line), which would correspond to periodic travelling wave solutions of (15)-(11) with $F'(\rho) = \rho$.

3.2.4 The role of transverse directions

To conclude, let us say a few words about the stability of planar heteroclinic/homoclinic waves in several space dimensions. By Fourier transform in the (hyper)plane where the wave is constant, we are left with eigenvalue equations for operators $\mathscr{A}(\eta)$ (and $\mathscr{B}(\eta)$) parametrized by wave vectors $\eta \in \mathbb{R}^{d-1}$, which may be viewed as perturbations of the one-D operator $\mathscr{A}$ (and respectively $\mathscr{B}$). Let us recall that for heteroclinic waves $\mathscr{A}$ (and $\mathscr{B}$) has no unstable spectrum. It turns out that in several space dimensions, the wave vector η plays a 'stabilizing role', and that $\mathscr{A}(\eta)$ does not have any point spectrum outside the imaginary axis either, see [5]. On the contrary, transverse directions η 'destabilize' homoclinic waves. This was shown in [45], where a small unstable eigenvalue τ was found for arbitrary wave vectors η, and independently in [6] using the Evans function approach (the multi-D Evans function $D(\tau, \eta)$ being smooth on rays in $\mathbb{R}^+ \times \mathbb{R}^{d-1}$) and Rouché's theorem (similarly as in [49] for parabolic PDEs) to find a small unstable eigenvalue $\tau = O(\eta)$ for small wave vectors η.

Appendix

A.1 Derivation of the Euler–Korteweg system

A rather short way is the one followed by Rohde in [44], which we describe here for completeness. The idea is to find motions, in Eulerian coordinates, as critical points of the space-time Lagrangian

$$\frac{1}{2}\rho\,\|\mathbf{u}\|^2 \,-\, \mathscr{E}(\rho, \nabla\rho)$$

under two differential constraints, namely the conservation law

$$\partial_t \rho + \mathrm{div}(\rho\mathbf{u}) \,=\, 0 \qquad \text{(A.1)}$$

for the mass density, and the transport equation

$$\partial_t \mathbf{p} + (\mathbf{u}\cdot\nabla)\mathbf{p} \,=\, 0 \tag{A.2}$$

for the initial point positions. The existence of the field $\mathbf{p}$ implicitly assumes that the flow map $\boldsymbol{\chi}$ defined by $\partial_t\boldsymbol{\chi} \,=\, \mathbf{u}(\boldsymbol{\chi},t)$ and $\boldsymbol{\chi}(\boldsymbol{\xi},0)=\boldsymbol{\xi}$ for all $\boldsymbol{\xi}\in\mathbb{R}^d$ is global and nonsingular, in such a way that $\boldsymbol{\xi}=\mathbf{p}(\mathbf{x},t)$ is equivalent to $\mathbf{x}=\boldsymbol{\chi}(\boldsymbol{\xi},t)$. (We consider motions in the whole space $\mathbb{R}^d$ in order to avoid boundary conditions issues.) Then by adding a scalar unknown φ and a vector-valued unknown $\mathbf{q}$, we are left with looking for critical points of

$$L := \frac{1}{2}\rho\,\|\mathbf{u}\|^2 \,-\, \mathscr{E}(\rho,\nabla\rho) \,+\, \varphi\,(\partial_t\rho + \mathbf{u}\cdot\nabla\rho \,+\, \rho\,\mathrm{div}\mathbf{u}) \,-\, \rho\,\mathbf{q}\,(\partial_t\mathbf{p} + (\mathbf{u}\cdot\nabla)\mathbf{p})\,.$$

We recover of course the constraints (A.1) and (A.2), these equations being nothing but $\mathsf{E}_\varphi L = 0$ and $\mathsf{E}_{\mathbf{q}} L = 0$ (the latter provided that $\rho\neq 0$), and $\mathsf{E}_\rho L = 0$ gives

$$\partial_t\varphi \,+\, \mathbf{u}\cdot\nabla\varphi \,-\, \frac{1}{2}\|\mathbf{u}\|^2 \,+\, \mathsf{E}_\rho\mathscr{E} \,=\, 0\,, \tag{A.3}$$

while $\mathsf{E}_{\mathbf{u}} L = 0$ and $\mathsf{E}_{\mathbf{p}} L = 0$ give respectively

$$\mathbf{u} = \nabla\varphi + \sum_{j=1}^{d} q_j\nabla p_j\,,\quad \partial_t(\rho\,\mathbf{q}) \,+\, \sum_{j=1}^{d}\partial_{x_j}(\rho\,u_j\mathbf{q}) \,=\, 0\,.$$

Thanks to the mass conservation law (A.1), the latter can equivalently be rewritten as

$$\partial_t\mathbf{q} \,+\, (\mathbf{u}\cdot\nabla)\mathbf{q} \,=\, 0\,.$$

By some elementary algebra we can now eliminate φ, $\mathbf{p}$ and $\mathbf{q}$ from the equations above to retrieve the sought, second equation in the Euler–Korteweg system

$$\left\{\begin{array}{l} \partial_t\rho + \mathrm{div}(\rho\mathbf{u}) \,=\, 0\,, \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} \,+\, \nabla(\mathsf{E}_\rho\mathscr{E}) \,=\, 0\,. \end{array}\right. \tag{A.4}$$

(Note that if we omitted the constraint (A.2) we would just recover the equation for a *potential* velocity $\mathbf{u}=\nabla\varphi$, Eq. (A.3) then being Bernoulli's form of that equation.)

A.2 The Euler–Korteweg system in Clebsch coordinates

The following way of deriving the Euler–Korteweg equations is excerpted from [47]. We consider 3D velocity fields $\mathbf{u}$ admitting so-called Clebsch coordinates [21] (also see the book by Lamb[38, §167]),

$$\mathbf{u} \,=\, \nabla\varphi \,+\, \lambda\,\nabla\mu\,,$$

where $\nabla\varphi$ is clearly potential (and thus irrotational), and μ is a 'secondary' potential associated with a third unknown λ. The Euler–Korteweg system (A.4) turns out to be a by-product of the Euler–Lagrange equations for the Lagrangian

$$L := -\rho\,(\partial_t\varphi \,+\, \lambda\,\partial_t\mu) \,-\, \frac{1}{2}\rho\,\|\mathbf{u}\|^2 \,-\, \mathscr{E}(\rho,\nabla\rho)\,.$$

Indeed, we have

$$
\begin{aligned}
\mathsf{E}_\varphi L = 0 \quad &\Leftrightarrow \quad \partial_t \rho \,+\, \operatorname{div}(\rho\,\mathbf{u}) \;=\; 0\,,\\
\mathsf{E}_\mu L = 0 \quad &\Leftrightarrow \quad \partial_t(\rho\lambda) \,+\, \operatorname{div}(\rho\,\lambda\,\mathbf{u}) \;=\; 0\,,\\
\mathsf{E}_\lambda L = 0 \quad &\Leftrightarrow \quad \rho\partial_t\mu \,+\, \rho\,\mathbf{u}\cdot\nabla\mu \;=\; 0\,,\\
\mathsf{E}_\rho L = 0 \quad &\Leftrightarrow \quad \partial_t\varphi \,+\, \lambda\,\partial_t\mu \,+\, \tfrac{1}{2}\|\mathbf{u}\|^2 \,+\, \mathsf{E}_\rho\mathscr{E} \;=\; 0\,.
\end{aligned}
$$

Differentiating the last equation and using successively the identities

$$
\nabla\left(\frac{1}{2}\|\mathbf{u}\|^2\right) = \mathbf{u}\wedge(\nabla\wedge\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u}, \quad \mathbf{u}\wedge(\nabla\wedge\mathbf{u}) = \mathbf{u}\wedge(\nabla\lambda\wedge\nabla\mu) = (\mathbf{u}\cdot\nabla\mu) - (\mathbf{u}\cdot\nabla\lambda)\nabla\mu\,,
$$

together with the first three equations $\mathsf{E}_\varphi L = 0$, $\mathsf{E}_\mu L = 0$, $\mathsf{E}_\rho L = 0$, we recover the velocity equation as expressed in (A.4). Now, we may introduce the Hamiltonian

$$
\mathscr{H} := \varphi_t\,\frac{\partial L}{\partial\varphi_t} \,+\, \mu_t\,\frac{\partial L}{\partial\mu_t} \,-\, L \,=\, \frac{1}{2}\,\rho\,\|\mathbf{u}\|^2 \,+\, \mathscr{E}\,,
$$

to be seen (at least formally) as a function of $(\rho = -\frac{\partial L}{\partial\varphi_t}, \Lambda := \rho\lambda = -\frac{\partial L}{\partial\mu_t}, \varphi, \mu, \nabla\varphi, \nabla\mu)$. Then the Euler–Lagrange system $\delta L = 0$ is equivalent to

$$
\left\{
\begin{aligned}
&\partial_t\rho = \mathsf{E}_\varphi\mathscr{H}\,,\\
&\partial_t\Lambda = \mathsf{E}_\mu\mathscr{H}\,,\\
&\partial_t\varphi = -\mathsf{E}_\rho\mathscr{H}\,,\\
&\partial_t\mu = -\mathsf{E}_\Lambda\mathscr{H}\,,
\end{aligned}
\right.
$$

or equivalently,

$$
\partial_t \begin{pmatrix} \rho \\ \Lambda \\ \varphi \\ \mu \end{pmatrix} = \mathbf{J}\,\delta\mathscr{H}[\rho,\Lambda,\varphi,\mu]\,, \quad \mathbf{J} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
$$

This is a Hamiltonian formulation of the Euler–Korteweg system in “canonical” coordinates[11].

A.3 The role of Boussinesq’s moment convexity

Let us explain how (the most elementary part in) the Grillakis–Shatah–Strauss theory (Theorem 3.3 in [29]) applies to the Euler–Korteweg system, that is, why the strict convexity of Boussinesq’s moment implies that the unstable eigenvector of the Hessian of the constrained energy is transverse to the level sets of Benjamin’s impulse (and therefore that unstable direction is harmless regarding stability of the wave, which is Theorem 3.5 in

[11] The term *canonical* refers to the special form of $\mathbf{J}$.

[29]). For concreteness we consider the Euler–Korteweg system in Eulerian coordinates - the very same arguments would work in Lagrangian coordinates. Assume that for each σ we have a stationary solution $(\rho, u) = (R, U)(\xi)$ of

$$\left\{ \begin{array}{l} \partial_t \rho = -\partial_\xi (\mathsf{E}_u(\mathscr{H} - \sigma \mathscr{Q}))\,, \\ \partial_t u = -\partial_\xi (\mathsf{E}_\rho(\mathscr{H} - \sigma \mathscr{Q}))\,, \end{array} \right.$$

that is homoclinic to (ρ_∞, u_∞). We simply denote by M the function of σ defined by

$$\mathsf{M}(\sigma) = \int \mathscr{M}\,\mathrm{d}\xi\,, \quad \mathscr{M} = \mathscr{L}(R, U, R_\xi) - \mathscr{L}(\rho_\infty, u_\infty, 0)\,,$$

$$\mathscr{L} := \mathscr{H} - \sigma\,\mathscr{Q} - \mu_1\,\rho - \mu_2\,u\,, \quad \mu_1 := \frac{\partial \mathscr{E}}{\partial \rho}(\rho_\infty, 0) + \frac{j^2}{2\rho_\infty^2} - \tfrac{1}{2}\sigma^2\,, \quad \mu_2 := j := \rho_\infty\,(u_\infty - \sigma)\,.$$

Recalling that $\delta\mathscr{L} = 0$ at (R, U), we find that

$$\mathsf{M}'(\sigma) = -\int \left(\mathscr{Q} - \mathscr{Q}_\infty + (R - \rho_\infty)\,\frac{\partial \mu_1}{\partial \sigma} + (U - u_\infty)\,\frac{\partial \mu_2}{\partial \sigma} \right) \mathrm{d}\xi\,.$$

Noticing that

$$\frac{\partial \mu_1}{\partial \sigma} = -\,u_\infty\,,\ \frac{\partial \mu_2}{\partial \sigma} = -\,\rho_\infty\,,$$

we can simplify that expression into

$$\mathsf{M}'(\sigma) = -\int (R - \rho_\infty)\,(U - u_\infty)\,\mathrm{d}\xi\,,$$

hence

$$\mathsf{M}''(\sigma) = -\int ((R - \rho_\infty)\,U_\sigma + (U - u_\infty)\,R_\sigma\,)\,\mathrm{d}\xi\,,$$

where for simplicity we have denoted by R_σ and U_σ the derivatives of R and U with respect to σ. Furthermore, by differentiating the profile equations $\delta\mathscr{L} = 0$ we get

$$(\mathsf{Hess}\mathscr{H} - \sigma\,\mathsf{Hess}\mathscr{Q})(R, U) \begin{pmatrix} R_\sigma \\ U_\sigma \end{pmatrix} = \begin{pmatrix} U - u_\infty \\ R - \rho_\infty \end{pmatrix}.$$

Therefore, denoting by $\langle \cdot, \cdot \rangle$ the L^2 inner product, we see that

$$\mathsf{M}''(\sigma) = -\left\langle (R_\sigma, U_\sigma), \mathbf{B} \begin{pmatrix} R_\sigma \\ U_\sigma \end{pmatrix} \right\rangle, \quad \mathbf{B} := (\mathsf{Hess}\mathscr{H} - \sigma\,\mathsf{Hess}\mathscr{Q})(R, U)\,.$$

Consequently, if $\mathsf{M}''(\sigma) > 0$, the self-adjoint operator $\mathbf{B}$ has at least one negative eigenvalue. This we can also infer from a Sturm–Liouville argument. Indeed, we know by differentiation of $\delta\mathscr{L} = 0$ with respect to ξ that $\begin{pmatrix} R_\xi \\ U_\xi \end{pmatrix}$ is in the kernel of $\mathbf{B}$. Observing that R_ξ vanishes exactly once, and eliminating u from the eigenvalue equations

$$\mathbf{B} \begin{pmatrix} \rho \\ u \end{pmatrix} = \tau \begin{pmatrix} \rho \\ u \end{pmatrix},$$

we can invoke Sturm–Liouville theory to justify that $\mathbf{B}$ has exactly one negative eigenvalue. Besides, the spectrum of the asymptotic operator

$$\mathbf{B}_\infty := (\mathsf{Hess}\mathscr{H} - \sigma\,\mathsf{Hess}\mathscr{Q})(\rho_\infty, u_\infty)$$

is found to be positive and bounded away from zero because the endstate (ρ_∞, u_∞) is an hyperbolic fixed point of the profile ODEs ($\delta\mathscr{L} = 0$). As a consequence, the essential spectrum of $\mathbf{B}$ is positive and bounded away from zero. Now the explicit formula

$$\mathbf{B}\begin{pmatrix} R_\sigma \\ U_\sigma \end{pmatrix} = \begin{pmatrix} U - u_\infty \\ R - \rho_\infty \end{pmatrix}$$

found above says in fact much more, and enables to us to show that $\mathbf{B}$ is positive definite on the space orthogonal to the plane spanned by $\begin{pmatrix} R_\xi \\ U_\xi \end{pmatrix}$ and $\begin{pmatrix} U - u_\infty \\ R - \rho_\infty \end{pmatrix}$. Indeed, let us denote for simplicity

$$\mathbf{U}_\xi := \begin{pmatrix} R_\xi \\ U_\xi \end{pmatrix}, \quad \mathbf{U}_\sigma := \begin{pmatrix} R_\sigma \\ U_\sigma \end{pmatrix}, \text{ and } \mathbf{Q} := \begin{pmatrix} U - u_\infty \\ R - \rho_\infty \end{pmatrix}$$

We know that $\mathbf{B}$ is self-adjoint,

$$\mathbf{B}\mathbf{U}_\xi = 0\,,\ \mathbf{B}\mathbf{U}_\sigma = \mathbf{Q}\,,\ \text{and}\ \langle \mathbf{Q}, \mathbf{U}_\sigma\rangle < 0\,,$$

and by our considerations on $\mathbf{B}$'s spectrum, $\mathbf{B}$ is positive definite on the space orthogonal to the plane spanned by $\mathbf{U}_\xi$ and $\mathbf{X}$, say a unitary eigenvector of $\mathbf{B}$ associated with its only negative eigenvalue λ. Let us denote by Π the orthogonal projection onto $(\mathrm{span}(\mathbf{U}_\xi, \mathbf{X}))^\perp$, and take $\mathbf{Y} \in (\mathrm{span}(\mathbf{U}_\xi, \mathbf{Q}))^\perp$. We have

$$\begin{array}{llll} \lambda\,\langle \mathbf{U}_\sigma, \mathbf{X}\rangle^2 & +\,\langle \mathbf{B}\Pi(\mathbf{U}_\sigma), \Pi(\mathbf{U}_\sigma)\rangle & = \langle \mathbf{B}\mathbf{U}_\sigma, \mathbf{U}_\sigma\rangle & < 0\,, \\ \lambda\,\langle \mathbf{U}_\sigma, \mathbf{X}\rangle\,\langle \mathbf{Y}, \mathbf{X}\rangle & +\,\langle \mathbf{B}\Pi(\mathbf{U}_\sigma), \Pi(\mathbf{Y})\rangle & = \langle \mathbf{B}\mathbf{U}_\sigma, \mathbf{Y}\rangle & = 0\,, \\ \lambda\,\langle \mathbf{Y}, \mathbf{X}\rangle^2 & +\,\langle \mathbf{B}\Pi(\mathbf{Y}), \Pi(\mathbf{Y})\rangle & = \langle \mathbf{B}\mathbf{Y}, \mathbf{Y}\rangle\,. & \end{array}$$

Recalling that

$$\langle \mathbf{B}\Pi(\mathbf{U}_\sigma), \Pi(\mathbf{U}_\sigma)\rangle > 0\,,$$

and using the Cauchy-Schwarz inequality, we thus see that

$$\begin{aligned} \langle \mathbf{B}\mathbf{Y}, \mathbf{Y}\rangle \ &\geq \lambda\,\langle \mathbf{Y}, \mathbf{X}\rangle^2 + \frac{\langle \mathbf{B}\Pi(\mathbf{U}_\sigma), \Pi(\mathbf{Y})\rangle^2}{\langle \mathbf{B}\Pi(\mathbf{U}_\sigma), \Pi(\mathbf{U}_\sigma)\rangle} \\ &= \lambda\,\langle \mathbf{Y}, \mathbf{X}\rangle^2 \left(1 + \lambda\,\frac{\langle \mathbf{U}_\sigma, \mathbf{X}\rangle^2}{\langle \mathbf{B}\Pi(\mathbf{U}_\sigma), \Pi(\mathbf{U}_\sigma)\rangle}\right) > 0\,. \end{aligned}$$

References

[1] J.C. Alexander, R. Gardner, and C.K.R.T. Jones. A topological invariant arising in the stability analysis of travelling waves. *J. Reine Angew. Math.*, 410:167–212, 1990.

[2] V. I. Arnold and B. A. Khesin. *Topological methods in hydrodynamics*, volume 125 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[3] T. B. Benjamin. The stability of solitary waves. *Proc. Roy. Soc. (London) Ser. A*, 328:153–183, 1972.

[4] T. B. Benjamin. Impulse, flow force and variational principles. *IMA J. Appl. Math.*, 32(1-3):3–68, 1984.

[5] S. Benzoni-Gavage. Linear stability of propagating phase boundaries in capillary fluids. *Phys. D*, 155(3-4):235–273, 2001.

[6] S. Benzoni-Gavage. Spectral transverse instability of solitary waves in Korteweg fluids. *J. Math. Anal. Appl.*, 361(2):338–357, 2010.

[7] S. Benzoni-Gavage, R. Danchin, and S. Descombes. On the well-posedness for the Euler-Korteweg model in several space dimensions. *Indiana Univ. Math. J.*, 56:1499–1579, 2007.

[8] S. Benzoni-Gavage, R. Danchin, S. Descombes, and D. Jamet. Structure of Korteweg models and stability of diffuse interfaces. *Interfaces Free Boundaries*, 7:371–414, 2005.

[9] S. Benzoni-Gavage, R. Danchin, S. Descombes, and D. Jamet. Stability issues in the Euler-Korteweg model. In *Joint Summer Research Conference* Control methods in PDE-Dynamical Systems, volume 426 of *Contemporary Mathematics*, pages 103–127. AMS, 2007.

[10] S. Benzoni-Gavage, Raphaël Danchin, and Stéphane Descombes. Well-posedness of one-dimensional Korteweg models. *Electronic J. Diff. Equations*, 2006(59):1–35, 2006.

[11] S. Benzoni-Gavage, P. Noble, and L. M. Rodrigues. On whitham's modulated equations for the Euler-Korteweg system. In preparation, 2012.

[12] Victor L. Berdichevsky. *Variational principles of continuum mechanics. I.* Interaction of Mechanics and Mathematics. Springer-Verlag, Berlin, 2009. Fundamentals.

[13] F. Bethuel, G. Orlandi, and D. Smets. Vortex rings for the Gross-Pitaevskii equation. *J. Eur. Math. Soc. (JEMS)*, 6(1):17–94, 2004.

[14] J. L. Bona and R. L. Sachs. Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation. *Comm. Math. Phys.*, 118(1):15–29, 1988.

[15] J. L. Bona, P. E. Souganidis, and W. A. Strauss. Stability and instability of solitary waves of Korteweg-de Vries type. *Proc. Roy. Soc. London Ser. A*, 411(1841):395–412, 1987.

[16] J. Boussinesq. Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *J. Math. Pures Appl.*, 17(2):55–108, 1872.

[17] D. Bresch, B. Desjardins, and C.-K. Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations*, 28(3-4):843–868, 2003.

[18] D. Bresch and P. Noble. Mathematical justification of a shallow water model. *Methods Appl. Anal.*, 14(2):87–117, 2007.

[19] T. J. Bridges and G. Derks. Constructing the symplectic Evans matrix using maximally analytic individual vectors. *Proc. Roy. Soc. Edinburgh Sect. A*, 133(3):505–526, 2003.

[20] D. Chiron. Travelling waves for the nonlinear Schrödinger equation with general nonlinearity in dimension one. Preprint, 2011.

[21] Alfred Clebsch. *J. Reine Angew. Math.*, 54 & 56(1):1–10, 1859.

[22] Bernard Deconinck and Todd Kapitula. On the orbital (in)stability of spatially periodic stationary solutions of generalized Korteweg-de Vries equations. Preprint, 2010.

[23] Nelson Dunford and Jacob T. Schwartz. *Linear operators. Part III.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Spectral operators, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1971 original, A Wiley-Interscience Publication.

[24] Thierry Gallay and Mariana Hărăgus. Orbital stability of periodic waves for the nonlinear Schrödinger equation. *J. Dynam. Differential Equations*, 19(4):825–865, 2007.

[25] R. A. Gardner. On the structure of the spectra of periodic travelling waves. *J. Math. Pures Appl. (9)*, 72(5):415–439, 1993.

[26] R.A. Gardner and K. Zumbrun. The gap lemma and geometric criteria for instability of viscous shock profiles. *Comm. Pure Appl. Math.*, 51(7):797–855, 1998.

[27] Robert A. Gardner. Spectral analysis of long wavelength periodic waves and applications. *J. Reine Angew. Math.*, 491:149–181, 1997.

[28] Sergey L. Gavrilyuk and Denis Serre. A model of a plug-chain system near the thermodynamic critical point: connection with the Korteweg theory of capillarity and modulation equations. In *Waves in liquid/gas and liquid/vapour two-phase systems (Kyoto, 1994)*, volume 31 of *Fluid Mech. Appl.*, pages 419–428. Kluwer Acad. Publ., Dordrecht, 1995.

[29] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.*, 74(1):160–197, 1987.

[30] Manoussos Grillakis. Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system. *Comm. Pure Appl. Math.*, 43(3):299–333, 1990.

[31] Mariana Hărăguş and Todd Kapitula. On the spectra of periodic waves for infinite-dimensional Hamiltonian systems. *Phys. D*, 237(20):2649–2671, 2008.

[32] J. Höwing. Stability of large- and small-amplitude solitary waves in the generalized Korteweg-de Vries and Euler-Korteweg / Boussinesq equations. Preprint, 2011.

[33] Mathew A. Johnson, Kevin Zumbrun, and Jared C. Bronski. On the modulation equations and stability of periodic generalized Korteweg-de Vries waves via Bloch decompositions. *Phys. D*, 239(23-24):2057–2065, 2010.

[34] T. Kapitula and B. Sandstede. Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations. *Phys. D*, 124(1-3):58–103, 1998.

[35] Todd Kapitula. The Krein signature, Krein eigenvalues, and the Krein oscillation theorem. *Indiana Univ. Math. J.*, 59(4):1245–1275, 2010.

[36] Todd Kapitula, Panayotis G. Kevrekidis, and Björn Sandstede. Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems. *Phys. D*, 195(3-4):263–282, 2004.

[37] R. Kollár and R. L. Pego. Spectral stability of vortices in two-dimensional Bose–Einstein condensates via the Evans function and Krein signature. Preprint, 2010.

[38] Horace Lamb. *Hydrodynamics*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, sixth edition, 1993. With a foreword by R. A. Caflisch [Russel E. Caflisch].

[39] R. S. MacKay. Stability of equilibria of Hamiltonian systems. In *Nonlinear phenomena and chaos (Malvern, 1985)*, Malvern Phys. Ser., pages 254–270. Hilger, Bristol, 1986.

[40] M. Oh and K. Zumbrun. Stability of periodic solutions of conservation laws with viscosity: analysis of the Evans function. *Arch. Ration. Mech. Anal.*, 166(2):99–166, 2003.

[41] M. Oh and K. Zumbrun. Stability of periodic solutions of conservation laws with viscosity: pointwise bounds on the Green function. *Arch. Ration. Mech. Anal.*, 166(2):167–196, 2003.

[42] R.L. Pego and M.I. Weinstein. Eigenvalues, and instabilities of solitary waves. *Philos. Trans. Roy. Soc. London Ser. A*, 340(1656):47–94, 1992.

[43] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

[44] Christian Rohde. On local and non-local Navier-Stokes-Korteweg systems for liquid-vapour phase transitions. *ZAMM Z. Angew. Math. Mech.*, 85(12):839–857, 2005.

[45] F. Rousset and N. Tzvetkov. A simple criterion of transverse linear instability for solitary waves. *Math. Res. Lett.*, 17(1):157–169, 2010.

[46] Denis Serre. Spectral stability of periodic solutions of viscous conservation laws: large wavelength analysis. *Comm. Partial Differential Equations*, 30(1-3):259–282, 2005.

[47] W. van Saarloos, D. Bedeaux, and P. Mazur. Hydrodynamics for an ideal fluid: Hamiltonian formalism and Liouville equation. *Phys. A*, 107(1):109–125, 1981.

[48] G. B. Whitham. *Linear and nonlinear waves.* John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication.

[49] K. Zumbrun and D. Serre. Viscous and inviscid stability of multidimensional planar shock fronts. *Indiana Univ. Math. J.*, 48(3):937–992, 1999.

[50] Kevin Zumbrun. Dynamical stability of phase transitions in the p-system with viscosity-capillarity. *SIAM J. Appl. Math.*, 60(6):1913–1924 (electronic), 2000.

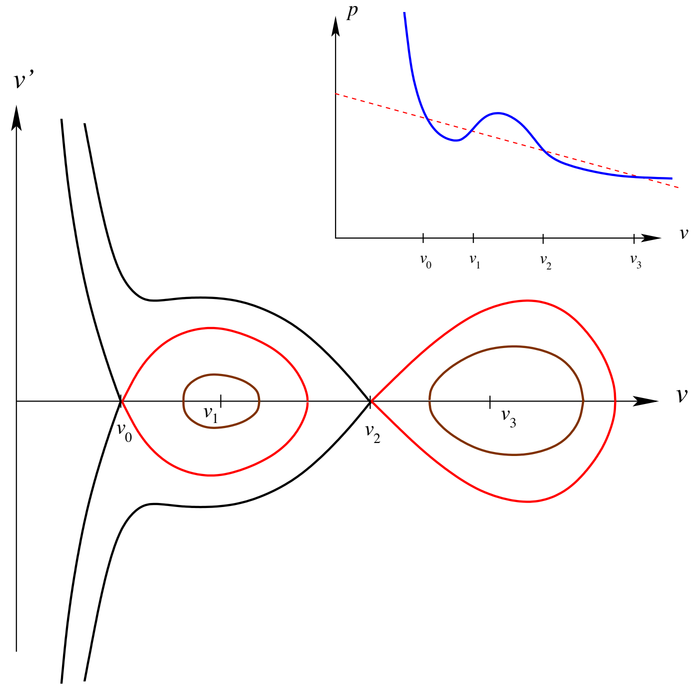

Figure 2: A nonconvex pressure law and the associated phase portrait for capillary profiles. (Observe that equal area rule yields two solitons.)

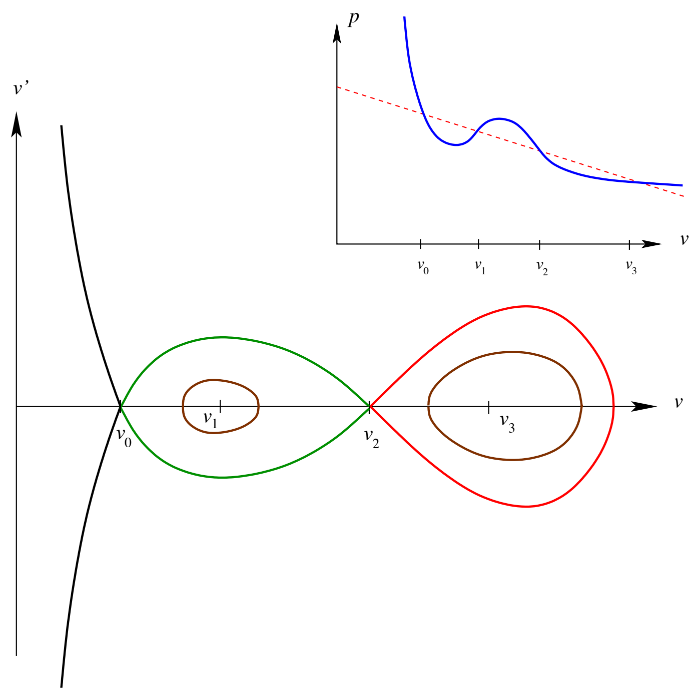

Figure 3: A nonconvex pressure law and the associated phase portrait for capillary profiles. (Observe that equal area rule yields one heteroclinic orbit and one soliton.)

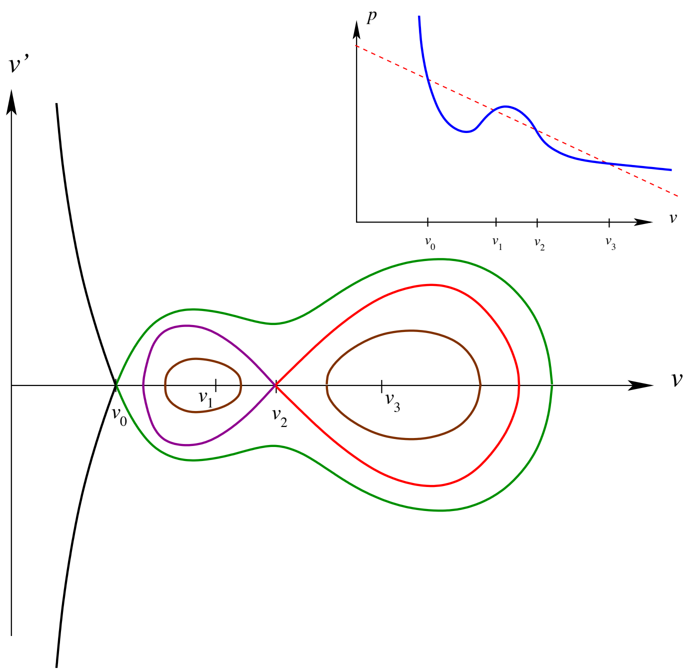

Figure 4: A nonconvex pressure law and the associated phase portrait for capillary profiles. (Observe that equal area rule yields three solitons.)